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Analytical Solution and Verification of Diffusion Effect in SSFP*

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Abstract

Assuming that the SSFP magnetization response maintains a steady state which is periodic in the presence of diffusion, we can solve for the diffusion effect in such sequences. Formulating a Fourier series decomposition solution to the Bloch–Torrey equation and imposing the steady-state condition, analytical expressions describing the signal decay due to diffusion are developed. Magnetization responses for any system and sequence parameters can then be obtained. Also, sensitivity to b factor changes is quite different than standard diffusion measurement techniques. Assumptions made in the solution are verified via finite difference solutions and simulations of the Bloch–Torrey equation.

INTRODUCTION

Steady-state free precession (SSFP) sequences involve the application of a stream of rf pulses at a frequency TR^{-1} . When $\text{TR} \ll T_2$, and in the presence of an equivalent gradient waveform each TR, the magnetization response is a longitudinal and transverse steady state which is actually a dynamic equilibrium with time period TR. SSFP was first described by Carr (1) and analytical solutions describing the response for static spins were obtained later (2,3). These solutions show that the magnetization response is a complex function of TR, relaxation times, rf tip angle, and the precession angle ψ of a spin between rf pulses. In the presence of a gradient, the SSFP response is spatially periodic since ψ is modulo 2π . The spatial wavelength, λ , of the response is then described as the distance one has to move in the gradient to achieve a phase shift in ψ of 2π ;

$$\lambda = \frac{1}{\gamma F(\text{TR})}; \quad F(t) = \int_0^t G(t') dt' . \quad [1]$$

SSFP sequences are widely used for imaging. A number of gradient-refocused versions exist which refocus the FID (4), the echo (5), or both. In addition, another class of SSFP methods where every n th pulse is missing (6,7), also known as missing pulse SSFP or MP-SSFP, allows the acquisition of a rf-refocused signal at the time of the missing rf pulse, thereby reducing the sensitivity to T_2^* of the gradient-refocused methods.

The unique characteristics of the SSFP magnetization response have been exploited to study flow (8). The technique is especially sensitive to flow in the millimeter per second range which makes it suitable for perfusion studies. Recently, the perfusion sensitivity of MP-SSFP has been evaluated in rabbit kidneys *in vitro* (10). The dependence of the SSFP signal on flow is related to the fact that static spins develop a spatially periodic response and that the time it takes for spins to establish a steady state is on the order of T_1 . The steady state which a flowing spin is trying to establish constantly changes as it traverses a distance equal to λ . Because of the time constant associated with establishing the steady state, the response of a moving spin

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becomes scrambled and is reduced from the average response of a static spin. The amount of signal decay is dependent on how far a spin moves relative to the spatial wavelength. This effect can be quantified with a dimensionless dephasing parameter (9),

$$\frac{\Delta u}{\lambda} = \frac{vTR}{\lambda}, \quad [2]$$

where Δu is the distance moved during the interpulse interval TR and v is the velocity of the moving spin. By decreasing the spatial wavelength the sensitivity to slow flow can be increased and significant signal decay can be achieved. From Eq. [1] it is seen that this is achieved by increasing the area under the gradient waveform. This adjustable sensitivity allows SSFP to study flow in the centimeter per second range down to flow in the perfusion range.

On the basis of the work of MP-SSFP in the kidney (10), we feel that SSFP can be a useful technique to qualitatively measure perfusion in some biological tissues. Tissues which will not be able to be evaluated are ones that have a bulk macroscopic motion associated with them, such as the heart or tissues connected to respiratory motion. If one is going to use the relative attenuation of the SSFP signal as a measure of perfusion, then it is important to know what the other sources of possible signal decay are. Thus the motivation for this work is to quantitatively evaluate the effect of diffusion on SSFP.

DIFFUSION SENSITIVITY

This section outlines our solution of the Bloch–Torrey equation in the SSFP context. Kaiser *et al.* (11) first proposed the Fourier series approach to the SSFP diffusion problem for the case of a constant gradient. Using an approximation that the signal consisted of only primary spin echoes and stimulated echoes, Merboldt *et al.* (12) performed an analysis of experimental data to explain the effect. Le Bihan *et al.* (13) approximated the SSFP diffusion effect by substituting a diffusion-dependent “effective T2” into the $D = 0$ or Freeman and Hill solution for the SSFP response. Wu and Buxton (14) have extended the Kaiser analysis for the case of a single pulsed gradient. The solution outlined here, which we label the Fourier series decomposition (FSD), takes a more general approach to the problem than the Kaiser analysis by accommodating any gradient waveform and assumes less a priori about the form of the response (15). The results are verified with Bloch–Torrey simulations. We base the FSD solution on the assumptions that the SSFP response in the presence of diffusion maintains some of its characteristics from the $D = 0$ solution, namely,

1. magnetization reaches a steady state;
2. magnetization response is periodic.

Based on our assumptions we assume a Fourier series solution to the Bloch–Torrey equation,

$$m_+(x, t) = M_x + iM_y = e^{-i\gamma F(t)x} \sum_{-\infty}^{+\infty} a_n(t) e^{-i\gamma(n/\lambda)x} \quad [3a]$$

$$M_z(x, t) = \sum_{-\infty}^{+\infty} b_n(t) e^{-i\gamma(n/\lambda)x}, \quad [3b]$$

where the exponential outside of the summation for the transverse magnetization represents the gradient precession. After substitution into Bloch–Torrey, we obtain a solution for the Fourier coefficients in the absence of rf pulses, (i.e., during the interpulse interval),

$$a_n(t) = a_n(0^+) e^{-g_n(t)} \quad [4a]$$

$$b_n(t) = b_n(0^+) e^{-p_n(t)}, \quad [4b]$$

where 0^+ is the time immediately following the rf pulse and

$$g_n(t) = \int_0^t \left\{ \frac{1}{T_2} + \gamma^2 D \left(F(t') + nF(TR) \right)^2 dt' \right\} \quad [5a]$$

$$p_n(t) = \int_0^t \left\{ \frac{1}{T_1} + \gamma^2 D n F(TR)^2 dt' \right\}. \quad [5b]$$

Now we include the rf pulses by treating them as instantaneous rotations, and we impose the steady state condition

$$m_+(x, t) = m_+(x, t + TR) \quad [6a]$$

$$M_z(x, t) = M_z(x, t + TR), \quad [6b]$$

where we let $t = 0^+$. A recursion relation is obtained among the transverse Fourier coefficients

$$a_n(t) = C_n a_{n-1}(t) + C_n' a_{-(n+1)}^*(t), \quad [7]$$

where

$$C_n = \frac{1}{2} \left\{ 1 + \cos(\theta) - \frac{\sin^2(\theta) e^{-P_n}}{1 - e^{-P_n \cos(\theta)}} \right\} e^{-g_{n-1}} \quad [8a]$$

$$C_n' = \frac{1}{2} \left\{ 1 - \cos(\theta) + \frac{\sin^2(\theta) e^{-P_n}}{1 - e^{-P_n \cos(\theta)}} \right\} e^{-g_{-(n+1)}} \quad [8b]$$

θ is the rf tip angle and * denotes conjugation. If now we assume that as the coefficient index $|n|$ increases, the magnitude of coefficients become negligible, we obtain a system of linear equations. We can solve these equations and for specific cases obtain values for all transverse Fourier coefficients at once.

RESULTS

Having obtained the coefficients, the magnetization response for different system and sequence parameters can be calculated, both for the echo and for the FID signal. Figure 1 shows the transverse components of the magnetization, in the steady state, over one spatial wavelength. The response decreases as the diffusion coefficient increases. Figure 2 shows the predicted signal decay as a function of the b factor. The b factor in Fig. 2 is per TR but the signal attenuation for SSFP is influenced by the gradient waveform over several TRs depending on the values of T_2 and D . Using the b factor definition for a single TR period, Fig. 2 shows that the SSFP signal has greater sensitivity to b factor changes than the standard diffusion measuring techniques (i.e., Stejskal–Tanner). Fig. 2 also shows how the SSFP signal decays depending on how the b factor is changed. Increasing b by adjusting TR results in more decay than by changing just the gradient amplitude. Thus for the same value of b there are two curves showing the signal loss. This is plausible because if one looks at the $D = 0$ solution it shows that by changing TR both the spatial wavelength and the signal amplitude are affected. Changing gradient amplitude only affects spatial wavelength and not signal amplitude.

VERIFICATION

As stated in the formulation of the FSD solution, the assumptions that the magnetization response was steady and spatially periodic in the presence of diffusion were made. Using computer simulations we have verified that these assumptions are valid and that the FSD solution is correct. The first simulation assumed solely that the response is periodic and not

necessarily in a steady state. The magnetization is evolved in time for approximately 50 TR periods to determine if a steady state is achieved. The Bloch–Torrey solution obtained in the absence of rf (Eq. [4]) is used to evolve the magnetization between rf pulses. The results show that a steady state is achieved and the response is identical to that predicted by the FSD solution.

The second simulation was a finite difference solution of Bloch–Torrey applied to the SSFP sequence. Nothing was assumed a priori about the form of magnetization response. A standard classic explicit finite difference method was used (16,17). Figure 3 shows the finite difference grid, where h is the time resolution and k the spatial resolution. The boundary conditions are

$$M_{0,j}^{x,y,z} = (M_{1,j}^{x,y,z} + M_{-1,j}^{x,y,z}) / 2 \quad [9a]$$

$$M_{N,j}^{x,y,z} = (M_{N+1,j}^{x,y,z} + M_{N-1,j}^{x,y,z}) / 2. \quad [9b]$$

The finite difference equation, stated only for M_x , is

$$M_{i,j+1}^x = \gamma G \left(i - \frac{N}{2} \right) M_{i,j}^y + M_{i,j}^x \left(1 - \frac{h}{T_2} - \frac{2hD}{k^2} \right) + \frac{hD}{k^2} (M_{i+1,j}^x + M_{i-1,j}^x), \quad [10]$$

where i is the spatial index and j the time index. Stability criteria for the grid spacing were derived:

$$\frac{h}{T_2} + \frac{4hD}{k^2} + \gamma G h k \frac{N}{2} < 2. \quad [11]$$

This finite difference solution is propagated through time and space, again treating the rf pulses as instantaneous rotations. Typical values of h were 5–10 μ s and for k , 1–5 μ m. These values would give roughly 8000–16,000 time steps per TR for a system of 10,000 spins per centimeter. We obtain magnetization responses identical to those predicted analytically.

CONCLUSION

We have successfully developed a general case solution for the diffusion effect for SSFP type sequences. The solution is based on assumptions that the magnetization response in the presence of diffusion maintains certain characteristics from the $D = 0$ solution. Both these assumptions and the FSD solution have proven to be true through numerical simulations.

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Fig. 1. SSFP FID magnetization distributions obtained using FSD solution shown over one spatial wavelength (0.01 cm). (a) $D = 0.0$, (b) $D = 0.5$, (c) $D = 1.0$, (d) $D = 2.0$, (e) $D = 4.0 \times 10^{-5}$ cm²/s. $T_1 = 1.0$ s, $T_2 = 0.8$ s, $TR = 40$ ms, $\theta = 90^\circ$.

Fig. 2.

Signal decay due to diffusion as a function of b factor for FID signal. (a) Standard diffusion measuring techniques (Stejskal–Tanner, Hahn, etc.). (b) SSFP FSD solution with gradient amplitude increasing. (c) SSFP FSD solution with TR increasing. $T_1 = 1.0$ s, $T_2 = 0.8$ s.

Fig. 3.

Finite difference grid used showing computational node. Time resolution given by h , spatial resolution given by k . Space index denoted by i , time index denoted by j .