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An Analytical SMASH Procedure (ASP) for Sensitivity-Encoded MRI

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Abstract

The simultaneous acquisition of spatial harmonics (SMASH) method of imaging with detector arrays can reduce the number of phase-encoding steps, and MRI scan time several-fold. The original approach utilized numerical gradient-descent fitting with the coil sensitivity profiles to create a set of composite spatial harmonics to replace the phase-encoding steps. Here, an analytical approach for generating the harmonics is presented. A transform is derived to project the harmonics onto a set of sensitivity profiles. A sequence of Fourier, Hilbert, and inverse Fourier transform is then applied to analytically eliminate spatially dependent phase errors from the different coils while fully preserving the spatial-encoding. By combining the transform and phase correction, the original numerical image reconstruction method can be replaced by an analytical SMASH procedure (ASP). The approach also allows simulation of SMASH imaging, revealing a criterion for the ratio of the detector sensitivity profile width to the detector spacing that produces optimal harmonic generation. When detector geometry is suboptimal, a group of quasi-harmonics arises, which can be corrected and restored to pure harmonics. The simulation also reveals high-order harmonic modulation effects, and a demodulation procedure is presented that enables application of ASP to a large numbers of detectors. The method is demonstrated on a phantom and humans using a standard 4-channel phased-array MRI system.

Keywords

image reconstruction; phased-arrays MRI; SMASH; harmonics; phase-encoding; phase correction; ASP

The simultaneous acquisition of MRI signals with multiple detectors is useful for at least two purposes. One is to increase the signal-to-noise (SNR) and/or field-of-view (FOV), as has been well-documented and implemented on commercial MRI scanners (1–3). Another is to decrease the scan time by reducing the number of phase-encoding steps by substituting spatial information derived from the sensitivity profiles of the individual detectors (4–9). This is exemplified by the recent developments of the simultaneous acquisition of spatial harmonics (SMASH) imaging (8) and the sensitivity-encoded (SENSE) imaging methods (9). In principle, all of the phase-encoding steps used for conventional MRI could be eliminated by deploying a sufficiently large number of detectors. However, because of technical limitations that include the detector coupling and the need for small detector dimensions but high SNR in order to

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achieve high resolution, this approach has not been fully implemented (4,5). However, hybrid approaches that combine encoding using a small number of detectors with MRI gradient phase-encoding using a reduced number of steps, have been implemented to achieve several-fold reductions in scan-time (6–9). Such approaches are now practical for MRI because the decoupling of an array of detector coils and multichannel acquisition are mature technologies incorporated in commercial phased arrays (1).

The SMASH method recognizes the similarity between phase-encoding with MRI gradients and the composite spatial harmonic sensitivity inherent in the detectors. It utilizes a numerical fitting routine to generate a set of spatial harmonics from the sensitivity profile of a multi-channel array of MRI detectors, to achieve multifold reductions in gradient phase-encoding steps. Although this numerical approach was key to demonstrating the original SMASH concept, it does not take advantage of the underlying analytical relationship between the weighting factors for the composite harmonics, the FOV, the spacing of the detectors, the harmonic orders, and the sensitivity profiles of the detector coils (8).

In this paper we describe an analytical SMASH procedure (ASP). The method replaces the numerical fitting approach of SMASH with a new transform that generates a set of spatial harmonics corresponding to the image representation in k -space (10). The transform directly generates the complex weighting factors for the composite harmonics based on the FOV, and the spacing of the detectors. The Fourier transform (FT) of the detector sensitivity profiles provides the proper scaling factors among the different orders of the generated harmonics. Images can then be reconstructed by standard two-dimensional (2D) FT. In order to implement the transform, spatially dependent phase errors introduced by the individual detectors in the array must be removed from the raw data. By applying both FT and Hilbert transforms (HT) together, not only are these phase errors removed, but the spatial encoding, including the phase information essential for complete image restoration, is preserved. The analytical transform also provides a useful tool for simulating the SMASH method, which yields a criterion for the ratio of the sensitivity profile width to the detector spacing that produces the maximum number of harmonics for a given number of detectors, and guidelines to deal with defective harmonics and high-order harmonic modulation. The generation of harmonics from the sensitivity profiles of a standard 4-channel phased-array using the transform, and image reconstruction with the ASP method, are demonstrated on phantom and human 1.5 T MRI data.

THEORY

The MRI Signal From an Array of Detectors

Consider an array of N detectors lined-up along the y -axis with spacing d . The k -space MRI signal from the n^{th} detector for a selected slice is:

$$s_n(k_x, k_y) = \sum_x \sum_y w(x, y) e^{i(k_x x + k_y y)} f_n(x, y) e^{i\varphi_n(x, y)}, \quad [1]$$

where x and y are spatial coordinates, $w(x, y)$ represents the spin density distribution weighted by the relaxation times T_1 and T_2 , $f_n(x, y)$ is the sensitivity profile of each individual detector, and $\varphi_n(x, y)$ represents phase error introduced by each detector

The spatial encoding factor, $e^{i(k_x x + k_y y)}$, includes both frequency-encoding and phase-encoding. If the frequency-encoding gradient is G_x and the data acquisition time variable is t , then the frequency-encoding term is $e^{ik_x x} = e^{i\gamma G_x x t}$, with $k_x = \gamma G_x t$. If the phase-encoding gradient increment is g_y , the phase-encoding step is m where $m = 0, 1, 2, \dots, M-1$ and M is the total number of phase-encoding steps, the phase-encoding gradient is $G_y = m g_y$, and the phase encoding gradient period is T , then the phase encoding term in Eq. [1] is $e^{ik_y y} = e^{i\gamma(m g_y) y T}$ with

$k_y = \gamma g_y(mT)$, where k_y is the spatial frequency in the y -direction. Here (mT) is equivalent to a pseudo-time variable which serves as the second time dimension in the 2D FT, and M is one of the factors that determine total scan time. However, the k_y can be encoded by other means. From the Larmor equation, $\gamma g_y T = 2\pi/Y$, so that $k_y = m(2\pi/Y)$, where Y is the FOV in the y -direction, thereby relating k_y to the spatial harmonic frequencies of order m , in units of $2\pi/Y$. Spatial harmonics of order m may thus be generated from the sensitivity profiles, $f_n(x, y)$, in Eq. [1]. Therefore, the $e^{ik_y y}$ term in Eq. [1] can result either from the phase-encoding gradients, applied serially, or from spatial sensitivity-encoding based on the sensitivity profiles $f_n(x, y)$ of an array detectors that receive the MRI signals in parallel fashion.

The sensitivity profile $f_n(x, y)$ of the n^{th} detector can be evaluated from the Biot-Savart law or other forms of Maxwell's equations (3,11–13), or by experimental measurements (8,14). As in the original SMASH method (8), we assume that the image plane is parallel to the coil array plane, and ignore the x -dependency of the sensitivity profiles to simplify the analytical transform, so that $f_n(x, y) = f_n(y)$. With this assumption, when all the coils in the phased array have substantially identical sensitivity profiles, the sensitivity profile of the n^{th} detector in Eq. [1] is $f_n(y) = f(y - nd)$. For real coils with finite x -dimension, this assumption may introduce slight errors when the FOV is large. Note that if the imaging plane is not parallel to the coil plane, a 3D sensitivity profile $f_n(x, y, z) = f_x(x_n) f_y(y - nd) f_z(z_n)$ can be introduced, where $f_y(y - nd)$ is used for sensitivity encoding as above, and $f_x(x_n)$ and $f_z(z_n)$ are derived from the orientation of the oblique plane. $f_x(x_n)$ and $f_z(z_n)$ are used to amend the results of the sensitivity encoding from $f_y(y - nd)$. Even so, as with SMASH, the image plane that is perpendicular to both the coil plane and the phase encoding direction, cannot be encoded. The problem can be overcome by changing the coil orientation.

The phase errors, $\varphi_n(x, y)$, imparted by each detector, arise from the difference in the phase of the transverse magnetization generated at a point in space, as it is detected by each of the coils in the array due to their different locations in space. These phase errors may cause serious problems for generating spatial harmonics if they are not dealt with properly. On the other hand, phase errors introduced by fixed or time-dependent acquisition delays, flow or motion, etc., will be the same for each detector coil so that their effect on the generation of harmonics will be insignificant, although they may cause image artifacts analogous to conventional imaging.

Our goals for imaging with ASP are twofold. The first is to develop an analytical transform between the phase-encoding, represented by the $e^{ik_y y}$ term in Eq. [1], and sensitivity profile, represented by $f(y - nd)$, so that a set of linear combinations of the latter can be used to replace a set of phase-encoding steps. The second goal is to eliminate the phase errors introduced by the detectors, $\varphi_n(x, y)$, while preserving spatial encoding, which allows the new transform to be implemented in real MRI systems.

An Analytical Transform for SMASH Imaging

A basic assumption of the SMASH method is that phase-encoding steps can be composed from a linear combination of the sensitivity profiles of the detectors in the array. This assumption can be written in the form:

$$e^{ik_y y} = \sum_{n=0}^{N-1} C(k_y, n) f(y - nd), \quad [2]$$

where the $C(k_y, n)$ form a set of weighting coefficients. FT of Eq. [2] yields a transform that can be used to calculate $C(k_y, n)$:

$$C(k_y, n) = \frac{e^{ind k_y}}{F(k_y)}, \quad [3]$$

where $F(k_y)$ is the FT of $f(y)$. Because $k_y = m(2\pi/Y)$, for a given FOV, Y , Eq. [3] can be rewritten as

$$C(m, n) = \frac{e^{i2\pi mn(d/Y)}}{F(m)}, \quad [4]$$

which again underscores the relationship between the weighting coefficient of composite harmonics, the FOV, the detector spacing, and the harmonic order for the n^{th} detector. The mathematical derivation of Eq. [3] and proof that it can be used for spatial encoding are presented in the Appendix. Although Eq. [3] is an explicit analytical expression of the weighting parameters of the linear combination in Eq. [2], the convergence of Eq. [2] is conditional, which can be attributed to the nonorthogonality of $f(y - nd)$, as will be demonstrated later by simulations.

In the situation where N detectors are used to encode the whole image along the y -direction, once $C(k_y, n)$ is determined, the composite k -space signal is, combining Eqs. [1] and [2]:

$$\begin{aligned} s(k_x, k_y) &= \sum_{n=0}^{N-1} C(k_y, n) s_n(k_x) \\ &= \sum_x \sum_y w(x, y) e^{ik_x x} \sum_{n=0}^{N-1} C(k_y, n) f(y - nd) \\ &= \sum_x \sum_y w(x, y) e^{i(k_x x + k_y y)}, \end{aligned} \quad [5]$$

assuming that the phase errors are corrected. Thus, the image signal is completely encoded in 2D k -space, just as if it were encoded using phase-encoding gradients. The image is reconstructed by 2D FT of Eq. [5]:

$$\begin{aligned} S(x, y) &= FT\{s(k_x, k_y)\} \\ &= \sum_x \sum_y w(x, y) \delta(x' - x) \delta(y' - y). \end{aligned} \quad [6]$$

Hybrid ASP Imaging

A practical way to implement ASP, given present technical limits on detector design and availability, is to combine partial gradient phase encoding and partial ASP encoding, to achieve complete spatial encoding and a several-fold reduction in scan time. The partial gradient phase-encoding is a decimation of the full phase encoding steps, with a down-sampling factor of $\beta < N$. The partial ASP encoding generates β spatial harmonics with decimated phase encoding data from the array of detectors. We define k_y^* as the spatial frequency for the partial gradient phase-encoding, k_y^{**} as the spatial frequency for the partial ASP encoding, and k_y is the spatial frequency for entire hybrid ASP.

The analytical transform, Eq. [3] or Eq. [4], can be applied to the hybrid ASP. We illustrate such an application with an example that is the same situation as we implemented on the scanner. The number of detectors is $N = 4$, and the phase encoding is decimated by factor $\beta = 2$. The zero- and first-order harmonics need to be generated with ASP. Therefore, the signals with decimated phase encoding are

$$s_n(k_x, k_y^*) = \sum_x \sum_y w(x, y) e^{i(k_x x + 2k_y^* y)} f(y - nd), \quad [7]$$

$$n = 0, 1, 2, 3$$

where $k_y^* = m^* (2\pi/Y)$, with $m^* = 0, 1, \dots, M/2 - 1$, assuming again that the phase errors are corrected. The two harmonics generated by the ASP are formed from the phased-array signals:

$$ss_1(k_x, k_y^*) = \sum_{n=0}^3 C(k_y^{**} [m^{**} = 0], n) s_n(k_x, k_y^*)$$

$$ss_2(k_x, k_y^*) = \sum_{n=0}^3 C(k_y^{**} [m^{**} = 1], n) s_n(k_x, k_y^*) \quad [8]$$

where $k_y^{**} = m^{**} (2\pi/Y)$. Substituting Eq. [7] and Eq. [2] into Eq. [8] results in

$$ss_1(k_x, k_y^*) = \sum_x \sum_y w(x, y) e^{i(k_x x + 2k_y^* y)}$$

$$= \sum_x \sum_y w(x, y) e^{i(k_x x + 2m^* (2\pi/Y) y)}$$

$$ss_2(k_x, k_y^*) = \sum_x \sum_y w(x, y) e^{i(k_x x + 2k_y^* y)} e^{i(2\pi/Y) y}$$

$$= \sum_x \sum_y w(x, y) e^{i(k_x x + (2m^* + 1)(2\pi/Y) y)}. \quad [9]$$

Note from the exponents that ss_1 corresponds to the even orders of k_y^* , while ss_2 corresponds to the odd orders. Thus, by interleaving ss_1 and ss_2 , Eq. [9] for the hybrid ASP method can be combined as:

$$s(k_x, k_y) = \sum_x \sum_y w(x, y) e^{i(k_x x + k_y y)} \quad [10]$$

where $k_y = m(2\pi/Y)$, $m = 0, 1, \dots, 2m^*, 2m^* + 1, \dots, M - 1$. This is equivalent to both conventional gradient phase-encoded MRI or fully encoded ASP as described in Eq. [5]. This theory is readily extended to other detector array sizes (N) and decimation factors β , by generating and combining additional composite signals in Eqs. [8] and [9] with weighting coefficients determined from Eq. [3].

A Fourier-Hilbert Transform (FT-HT) Method for Phase Correction

Thus far we have omitted the effects of phase errors, $\varphi_n(x, y)$, arising from the difference in the phase of the transverse magnetization as detected from each of the multiple detectors due to their different locations. Such space-related phase errors normally preclude use the transform in Eq. [3] to generate harmonics. Thus, removal of the phase errors becomes an inseparable part of ASP and effectively renders the detector coil sensitivity profiles translationally symmetric. We present a method using FT and HT to eliminate the phase distortion while retaining the spatial encoding information.

The FT of the partial phase-encoded raw data can be written as:

$$FT\{s_n(k_x, k_y^*)\} = S_n(x, y^*) = |S_n(x, y^*)| e^{i(\phi_n^{\min}(x, y^*) + \varphi_n(x, y^*))}. \quad [11]$$

Here, with FT, the spatial encoding information within the phase of the k -space signals $s_n(k_x, k_y^*)$ is completely mapped to the magnitude of the image domain signal $S_n(x, y^*)$. Therefore, the magnitude of the image $|S_n(x, y^*)|$ contains both the image signal intensity weighting and the spatial encoding, while the phase of the image contains both the minimum phase $\phi_n^{\min}(x, y^*)$ (15) and the phase error components, $\varphi_n(x, y^*)$. With both the minimum phase $\phi_n^{\min}(x, y^*)$ and the magnitude $|S_n(x, y^*)|$, the image can be inverse Fourier transformed (IFT) back to k -space free of the phase errors. However, if the phase errors are removed simply by IFT of the magnitude $|S_n(x, y^*)|$ only, additional distortions could be introduced in k -space despite being free of the original space-related phase errors. The difference between the FT of the minimum phase signal and the magnitude only signal is illustrated in Fig. 1. Our goal therefore is to eliminate the phase distortion while preserving the spatial encoding, the minimum phase, and the image weighting information in the image domain, then IFT the corrected image data back to k -space.

Because the minimum phase $\phi_n^{\min}(x, y^*)$ and the magnitude $|S_n(x, y^*)|$ are a HT pair, one can apply a HT to $|S_n(x, y^*)|$ to create $\phi_n^{\min}(x, y^*)$. The procedure to synchronize the phases of multiple channel signals can be summarized as

$$\begin{aligned} s_n(k_x, k_y^*) &\xrightarrow{FT} S_n(x, y^*), \\ |S_n(x, y^*)| &\xrightarrow{HT} S_n^h(x, y^*) = |S_n(x, y^*)| e^{i\phi_n^{\min}(x, y^*)}, \\ S_n^h(x, y^*) &\xrightarrow{IFT} s_n^h(k_x, k_y^*). \end{aligned} \quad [12]$$

The k -space signal $s_n^h(k_x, k_y^*)$ contains all of the essential spatial-encoding information, but is free of the phase distortion. It can now be used to generate harmonics

METHODS AND RESULTS

Harmonic Generation

The composite spatial harmonics are generated as follows. First, the transform in Eqs. [3] or [4] is used to calculate a set of weighting parameters, based on the harmonic order, coil index, coil spacing, FOV, and $1/F(k_y^*)$. Second, the FT-HT method in Eq. [12] is applied to the set of multichannel MRI data for phase correction. Third, linear combinations of the signals from the multiple detectors are generated with the proper weighting parameters to form the zero-order, first-order, etc., harmonics, as exemplified in Eq. [9]. The results of harmonic generation with and without the phase-correction on the data obtained from the phantom are shown in Fig. 2. Note that without phase coherence, the harmonic generation with ASP is not generally viable.

The generation of accurate harmonics is perhaps the most critical measure of success of the ASP method. If the harmonics are imperfect, alias artifacts due to the decimation may remain to a certain degree. With our analytical approach, the accuracy of the harmonics generated is determined primarily by three factors. The first is the extent to which the phase errors are removed and the phases of the multiple channel signals synchronized via the FT-HT method.

A detailed discussion is presented in the theory section. The second is the ratio of the width of $f(y)$ to the coil spacing. The optimal ratio is approximately unity, as will be shown in the simulation section. The third is the accuracy of determination of the sensitivity profile of the coil and the calculation of $1/F(k_y^{**})$.

Unlike the SMASH method which uses the sensitivity profiles $f(y - nd)$ to numerically derive the weighting parameters of the composite harmonics (8), ASP requires only $F(k_y)$, the FT of the $f(y)$, as the scaling factors among the generated harmonics. The hybrid ASP method only needs partial knowledge of $F(k_y)$, that which is $F(k_y^{**})$, where $k_y^{**} = m * (2\pi / \text{FOV})$ for $m^{**} = 0, 1, \dots, \beta - 1$, to determine the weighting coefficients $C(k_y^{**}, n)$, from Eqs. [3] or [4]. Thus, in our experiment with $\beta = 2$, only $F(k_y^{**}[m * = 0])$ and $F(k_y^{**}[m * = 1])$ are needed. Note also that when the dimensions of the detector coils in the x -direction are comparable to the extent of the object being imaged, the assumption that $1/F(k_y^{**})$ is independent of k_x may need modification to avoid artifacts. Figure 3 shows the dependency of $1/F(k_y^{**}[m * = 0])$ and $1/F(k_y^{**}[m * = 1])$ on k_x for the phantom data. Here the $1/F(k_y^{**})$ are constant across the major part of the sample and require some adjustment only near the edges.

Note that the theory can tolerate minor imperfections in the coil sensitivities that result in $f_n(y) \neq f(y - nd)$ for some detectors, provided that the underlying order of the composite harmonics is preserved. Large imperfections will be manifest in the images as aliasing artifacts. The assumption in the analysis that all of the sensitivity profiles are the same, when they may differ to some degree, may cause errors that alter the relative amplitudes of the spatial frequency components of the image.

The Protocol for ASP Imaging

The ASP method can be implemented with multi-coil arrays and various Fourier MRI pulse sequences whose phase-encoding gradient increment is increased β -fold, resulting in a β -fold reduction in the total number of phase-encoding steps. The decimated raw data from each channel of the detector array are saved for ASP reconstruction. The basic ASP protocol can be summarized as follows:

- a. Acquire a reference image or a sub-set of image to obtain $f(y)$ for calculating $1/F(k_y^{**})$;
- b. Acquire the partial gradient phase-encoded signals from the phased-array coils;
- c. Synchronize the phases of the signals with the FT-HT method (12);
- d. Generate the harmonics to replace the phase-encoding steps with Eqs. [4] and [8];
- e. Combine the harmonics by interleaving them to form a fully-encoded raw data, set using Eq. [9];
- f. Apply a 2D FT to the raw data to reconstruct the image.

Human Study

We demonstrate the methods and results of the hybrid ASP with in vivo MRI experiments performed on a GE Medical System (Milwaukee, WI) Signa 1.5 T scanner. A standard GE co-linear four-coil spinal phased-array with spacing $d = 10$ cm was used for detection. The full-width at half maximum (FWHM) of the sensitivity profile, σ , was determined directly from

MRI studies of a phantom, and was $\sigma = 2.4d$. Although this commercial GE phased-array coil does not satisfy the criterion for the optimal geometry required to maximize the available harmonics, it is nevertheless capable of generating two or three harmonics that can be used for spatial encoding. The details about optimal harmonic generation will be discussed in the simulation section. A gradient echo pulse sequence was therefore modified to decimate the phase-encoding steps with $\beta = 2$, reducing the total scan time by half. The raw data from each of the four receiver channels were saved separately for ASP reconstruction.

In vivo image data from the human legs were acquired and reconstructed by the ASP method. The decimated images from the four receive channels and the final ASP image are shown in Fig. 4. Alias artifacts due to decimation are clearly evident in Fig. 4a, b, c, and d, but are absent from the ASP image, Fig. 4e, which has the proper spatial encoding.

SIMULATIONS

The analytical transforms can be used to simulate spatial harmonic generation and evaluate conditions that introduce errors and distortion of the composite signals. Here, we assume that $f(y)$ in Eq. [2] is a Gaussian function with width σ equal to $\sqrt{2}$ of the Gaussian variance. Although real coil sensitivity profiles may differ some from Gaussian, the simulations serve as a useful guide for exploring harmonic generation. The generated harmonics are complex functions, which, if perfect, follow a circular trajectory when plotted in the complex plane. Harmonic distortions are thus manifested as deviations from circular trajectories in the complex plane. Note that the validity of the ASP for spatial encoding, presented in the Appendix B, does not depend on the precise form of $f(y)$, so that many other distributions can as well be used.

Our first simulation assumes $N = 9$ detectors. Based on the simulation results, we derive a criterion for the optimal detector geometry, assuming Gaussian sensitivity profiles, that provides the maximum number of useable harmonics for a given number of detectors. In addition, for the harmonics derived with non-optimal detector geometry, we distinguish some of them as quasi-harmonics, which can be restored and used for spatial encoding. Our second simulation assumes $N = 33$ detectors. A harmonic modulation phenomenon is identified for large m , and a method of demodulation presented. For brevity, only the $m \geq 0$ harmonics are displayed in the figures, the $m < 0$ harmonics are the same except for a π phase shift.

Generating Optimal Harmonics

Mathematically projecting a harmonic onto a set of sensitivity profiles $f(y - nd)$ is not always valid. One important factor affecting the validity of ASP is the ratio of the width of the sensitivity profile to the detector spacing. When the ratio σ/d is small, the harmonics are no longer smooth, as shown in Fig. 5. Based on the trajectories in the complex plane, we describe this as a concave distortion. On the other hand, when σ/d is large, the higher order harmonics are lost, the encoding resolution is compromised, as shown in Fig. 6. We define this as convex distortion.

There are two special cases where the harmonics are optimal. One is the unrealistic case where $\sigma \approx 0$, whereupon $f(y) \approx \delta(y)$, the composite signal becomes discrete, and the concave distortion disappears. In addition, because $F(k_y) \approx 1$, there is no convex distortion either. The other case is when $\sigma \approx d$, which is achievable by real detector arrays. Both concave and convex distortions are controlled sufficiently to avoid spoiling of both harmonic shape and resolution, as shown in Fig. 7. Consequently, the criterion $\sigma \approx d$ yields the maximum number of harmonics that can be usefully obtained for ASP image encoding, for a given number of detectors N with Gaussian-shaped field profiles. The Gaussian profile is an approximation for circular surface coils, and the precise criterion may vary with the sensitivity profile. Note that even with $\sigma \approx d$, Fig. 7 shows that the highest order harmonic, $m = \pm 3$, manifests some ellipticity. Such modulation,

not a convex distortion, will become more apparent when the number of detectors is large, as discussed later.

Quasi-harmonics

Failure to meet the criterion $\sigma \approx d$ does not mean that useful harmonics cannot be created. The diameter and spacing of phased-array coils currently in use is based on SNR considerations and the constraints imposed by geometrical decoupling between adjacent coils (1,3), which means that $\sigma > d$ in most cases. Therefore some convex distortion is perhaps inevitable when ASP encoding with such coils. The simulations reveal three regimes when operating under these conditions, as illustrated in Fig. 6. The first is where the harmonic order is low (e.g., $m = 0, 1$ in Fig. 6), wherein the composite signals have reasonable harmonic fidelity. The second is where the harmonic order is intermediate (e.g., $m = 2$). The composite signals retain their order, but exhibit distortion involving enlargement of both end lobes. The third regime is where the order is high (e.g., $m = 3$), so that the harmonic is completely lost from the composite signal, and cannot be created from the sensitivity profiles of the array. When N is large, the convex distortions can become very colorful, but still fall into these three regimes. Increasing σ/d reduces the number of useable harmonics (regime one), and increases the number of unusable harmonics in regime three.

We define the distorted composite signals in the second regime as quasi-harmonics. Knowing their envelopes, we can scale and restore the quasi-harmonics to pure harmonics, effectively increasing the number of useful harmonics that can be generated. For example, seven usable harmonics can be obtained from $N = 9$ detectors if $\sigma \approx d$, whereas only three harmonics are available when $\sigma = 2d$. If the quasi-harmonics are restored, the number of available harmonics becomes five. For N large, the simulation shows that even minor deviations from the $\sigma \approx d$ criterion significantly reduces the number of harmonics. The recovery of harmonics from quasi-harmonics is crucial for increasing ASP image resolution when the optimal geometry criterion is not satisfied.

High-Order Harmonic Modulation and Demodulation

Although experimentally challenging, simulations can investigate the performance of ASP for very large numbers of detectors, potentially revealing characteristics at high harmonic orders that are yet to be seen in experiments. Here we assume $N = 33$ detectors. With $\sigma = d$ the simulated harmonics for $m = 0, 1, \dots, 15$ are shown in Fig. 8.

Analysis of the frequency response of the high order harmonics by FT reveals the presence of another frequency component, offset by Δ from each harmonic frequency. The component is most obvious in the complex plots of the highest harmonics ($m = 12$ to 15), suggesting that the harmonics in Eq. [2] contain an additional component whose amplitude, a_m , increases with harmonic order:

$$e^{ik_y y} + a_m e^{-i(k_y + \Delta)y} = \sum_{n=0}^{N-1} C(k_y, n) f(y - nd). \quad [13]$$

Because a_m and Δ , can be accurately determined by frequency analysis (FT) of the harmonics, the composite signal can be demodulated simply by rearranging Eq. [13]:

$$e^{ik_y y} = \sum_{n=0}^{N-1} C(k_y, n) f(y - nd) - a_m e^{-i(k_y + \Delta)y}. \quad [14]$$

The effect of demodulation is demonstrated at the bottom of Fig. 8 for $m = 15$. The demodulation process renders the ASP approach suitable for large number of detectors.

CONCLUSIONS

We developed a complete analytical procedure for spatially encoded MRI using the sensitivity profiles of an array of detectors, including a method for correcting the phase errors of the signals arising from the different detectors. Compared to the original numerical SMASH method (8), our method has several advantages. First, the analytical transform (Eq. [4]) provides a quantitative relationship between the weighting coefficients of the composite signals, the detector geometry (spacing d), the sensitivity profile ($1/F(k_y)$), the image FOV, the harmonic order (m), and the detector index (n). This relationship establishes a theoretical foundation for parallel data acquisition and encoding with multiple detectors in MRI, and provides an efficient means of combining, processing and reconstructing the parallel MRI data. Second, the FT-HT method, which removes the space-related phase errors and analytically restores the phase coherence among the signals from the array of detectors, also relieves the potential burden and cost of using hardware to correlate the phases of the signals in multiple receivers, which has thus far been avoided in conventional phased-array MRI by using a root-of-the-sum-of-the-squares reconstruction (1), but not for MRS (16).

In addition, ASP provides a tool to simulate harmonic generation which can reveal important insights about the properties of the method. For example, simulation suggests criteria for the optimal detector geometry, which provides a new target for detector design that may enhance the performance of ASP. The simulations also demonstrate the capability of ASP to cope with high-order harmonic modulations that arise when using a large numbers of detectors. This allows the application of ASP to large detector arrays, paving the road for true parallel MRI and the achievement of manifold reductions in minimum MRI scan-time for fluoroscopy and other rapid dynamic studies that require high time resolution.

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APPENDIX

The Derivation of the Transform $C(k_y, n)$ in Eq. [3]

For any given k_y , $0 \leq k_y \leq 2\pi$, apply a discrete FT to both sides of Eq. [2]:

$$\begin{aligned}\delta(k'_y - k_y) &= \sum_y \left(\sum_n C(k_y, n) f(y - nd) e^{-ik'_y y} \right) \\ &= \sum_n C(k_y, n) \sum_y f(y - nd) e^{-ik'_y y} \\ &= \sum_n C(k_y, n) F(k'_y) e^{-indk'_y}.\end{aligned}$$

Rearranging the above equation, we have

$$\frac{\delta(k'_y - k_y)}{F(k'_y)} = \sum_n C(k_y, n) e^{-indk'_y},$$

which is a Fourier series expansion with coefficients,

$$\begin{aligned}C(k_y, n) &= \int_{k'_y=0}^{2\pi} \frac{e^{-indk'_y}}{F(k'_y)} \delta(k'_y - k_y) dk'_y \\ &= \frac{e^{indk_y}}{F(k_y)}.\end{aligned}$$

This is Eq. [3].

Validation That the Transform in Eq. [3] Can Be Used to Encode Space

For simplicity, consider spatial encoding of the signal in the y direction only. It is

$$s(k_y) = \sum_y \left(\sum_n C(k_y, n) f(y - nd) \right).$$

Applying the transform of Eq. [3], we have

$$s(k_y) = \sum_y \left(\sum_n \frac{e^{indk_y}}{F(k_y)} f(y - nd) \right).$$

Apply a discrete FT over $s(k_y)$ yields

$$S(y) = \sum_y \left(\sum_{k_y} \left(\sum_n \frac{e^{indk_y}}{F(k_y)} f(y - nd) \right) \right) e^{-ik_y y'}.$$

Let $y'' = -nd$, then

$$\begin{aligned} S(y) &= \sum_y \left(\sum_{k_y} \left(\frac{1}{F(k_y)} \sum_{f'} f(y' + y) e^{-ik_y y'} \right) \right) e^{-ik_y y'} \\ &= \sum_y \left(\sum_{k_y} e^{ik_y y} \right) e^{-ik_y y'} \\ &= \sum_y \delta(y - y'), \end{aligned}$$

which corresponds to y -spatial encoding, just as is in conventional gradient phase-encoding.

FIG 1.

Demonstration of the FT-HT method of phase recovery on a 1D image from 2 bottles, one with twice the signal of the other, which can be formed from the sum of four sinusoids. **a** is the original analytical signal, $p(y)$ (solid line, real; dashed line, imaginary). **b** is the projection, $P(f)$, which is the FT of $p(t)$. **c** is the magnitude of the original signal, $|p(y)|$, **d** is the FT of $|p(y)|$, showing distortion and loss of the image information compared with **b**. **e** is the minimum phase signal recovered from the magnitude $|p(y)|$ of **c** via HT (solid line, magnitude; dashed line, phase). **f** is the FT of this recovered signal, demonstrating full restoration of the image information from **d**, as compared to **b**. The horizontal axes are space (y) or spatial frequency (f). Vertical axes are in arbitrary units.

FIG 2.

Demonstration of the harmonic generation and the essential role of phase correction in the harmonic generation. **a**, **b**, **c**, and **d** are the images from four coils, the data along the central horizontal lines in the images are used to demonstrate the harmonic generation and phase correction in plots **e**–**p**. Plots **e**–**h** are the original Re and Im parts of the profiles without phase correction, and **i** and **j** are the Re and Im parts of the harmonics generated therefrom. Plot **i** is the zero-order harmonic, and **j** is the first-order harmonic. Both exhibit serious distortions of the composite harmonics due to the phase incoherence. Plots **k**–**n** are the Re and Im parts of the profiles after phase correction with the FT-HT method. Plots **o** and **p** are the Re and Im parts of the harmonics generated from **k**–**n**. Plot **o** is the zero-order harmonic, and **p** is the first-order harmonic. The harmonic character of **o** and **p** is obvious. The horizontal axes are y , and for plots, the vertical axes is image intensity in arbitrary units.

FIG 3.

The k_x dependency of $1/F(k_y^{**}[m^{**} = 0])$, (a), and $1/F(k_y^{**}[m^{**} = 1])$, (b), for data from a 28 cm diameter circular phantom using a phased-array. The vertical axes are in arbitrary units.

FIG 4.

a, b, c, and d are the coronal MR images of human legs from four receive channels with decimated gradient phase-encoding steps. The FOV is 40 cm, TR is 18 msec, data acquisition matrix is 256×128 , slice thickness is 5 mm. **e** is the hybrid ASP image reconstructed from raw data of images **a, b, c, and d**. The distortion and signal loss at the extreme edges of the scan plane are due to the non-linearity in the MRI gradients, and not the ASP encoding.

FIG 5.

Illustration of concave distortion of composite harmonics with $s/d = 0.5$ and 9 detector coils. The top row is the real (Re, solid line) and imaginary (Im, dash line) parts of the composite harmonics. The horizontal axes is y , and the vertical axes are in arbitrary units. The second row shows the trajectories of the composite harmonics in the complex-plane. The horizontal axes are the Re part, and the vertical axes are the Im part. m is the harmonic order.

FIG 6.

Illustration of convex distortion of composite harmonics with $s/d = 2$, 9 detector coils. The harmonic order is m . The top row shows the real (Re, solid line) and imaginary (Im, dash line) parts of the composite harmonics. The horizontal axes is y , and the vertical axes are in arbitrary units. The second row shows the trajectories of the composite harmonics in the complex-plane. The horizontal axes are the Re part, and the vertical axes are the Im part. When $m = 0, 1, 2$, these are quasi-harmonics. When $m = 3$, the composite signals are no longer harmonic.

FIG 7.

Illustration of optimal composite harmonic generation with $s/d = 1$, 9 detector coils. The harmonic order is m . The top row shows the real (Re, solid line) and imaginary (Im, dash line) parts of the composite harmonics. The horizontal axes is y , and the vertical axes are in arbitrary units. The second row shows the trajectories of the composite harmonics in the complex-plane. The horizontal axes are the Re part, and the vertical axes are the Im part.

FIG 8.

The composite harmonics of order m , for high order harmonic modulation and demodulation with 33 detector coils. Rows one, three, and five are real (Re, solid line) and imaginary (Im, dash lines) parts of the harmonics plotted with horizontal axes, y , and vertical axes in arbitrary units. Rows two, four, and six are the corresponding harmonic trajectories in the complex-plane plotted with horizontal axes as the Re part, and Im part on the vertical axes. The last pair of traces with $m = 15$ shows the results of the demodulation of the penultimate pair with $m = 15$.