Thermodynamic constraints on stochastic acceleration in compressional turbulence

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Recent observations in the solar wind have revealed an important phenomenon. In circumstances where stochastic acceleration is expected, a suprathermal tail on the distribution function is formed with a common spectral shape: the spectrum is a power law in particle speed with a spectral index of -5 . This common spectrum occurs in the quiet solar wind; in disturbed conditions downstream from shocks; and, in particular, throughout the heliosheath downstream from the termination shock of the solar wind currently being explored by Voyager 1. In this article, simple thermodynamic principles are applied to stochastic acceleration in compressional turbulence. The unique spectral index results when the entropy of the suprathermal tail has increased to the maximum allowable value. Relationships for the pressure in the suprathermal tail are also derived and found to be in agreement with observations. The results are shown to be consistent with the suprathermal tail arising from a cascade in energy, analogous to a turbulent cascade. The results may be applied broadly, because stochastic acceleration in compressional turbulence should be common in many astrophysical settings.

particle acceleration | space plasma physics | stochastic phenomena

Among the most significant observations in recent years in space physics has been the common spectral shape of suprathermal tails observed in the solar wind (1–4). The suprathermal tails occur at particle speeds of approximately twice the solar-wind flow speed and extend up to energies of a few megaelectronvolts per nucleon. In the frame of the solar wind, the tails are power laws with a spectral index of -5 when expressed as a distribution function in velocity space. This common spectral shape occurs in all conditions in the solar wind: in quiet times when there are no shocks present and even in disturbed conditions downstream from shocks. The spectral index of -5 occurs, with unprecedented constancy, throughout the heliosheath downstream from the termination shock of the solar wind, now being observed by Voyager 1 (5, 6).

Shown in Fig. 1 is a typical proton spectrum observed in the slow, quiet solar wind by the Solar Wind Ion Composition Spectrometer (SWICS) on the Ulysses spacecraft (7). The spectrum shown consists of three parts: the thermal distribution of the solar wind, interstellar pickup ions (interstellar neutral gas that is swept into the heliosphere, ionized by photoionization and charge exchange, and picked up by the solar wind), and the suprathermal tail. The spectral index of the suprathermal tail is -5 in the frame of the solar wind. The tail extends beyond the range of the SWICS and can be seen in other observations as extending up to a few megaelectronvolts per nucleon (e.g., ref. 3). The tail, with this spectral index, occurs in quiet times and also in the presence of shocks, particularly in the heliosheath (5).

The question that needs to be asked and answered is, why is the observed spectral index -5 , and what are the implications of this result? The quiet-time suprathermal tails in the solar wind, far from shocks, must result from stochastic acceleration, and yet most forms of stochastic acceleration do not naturally yield either power-law spectra or a particular spectral index (e.g., refs. 8–12). Suprathermal tails downstream from shocks are influ-

Fig. 1. Typical proton-velocity distribution function observed during times of quiet (least-disturbed) solar-wind conditions at ≈ 5 AU from the Sun. The spectrum is made up of three distinct proton populations: (i) the thermal bulk solar-wind particles that carry most of the mass (dotted curve), (ii) the interstellar pickup protons (solid curve) that have a flat spectrum that drops sharply at approximately twice the solar-wind speed (in the spacecraft frame), and (iii) the suprathermal tail (dashed curve) that has a power-law spectrum in the frame of the solar wind with a unique spectral index of -5 .

enced by acceleration at the shock front. Diffusive shock acceleration can yield power-law spectra but not, under all circumstances, a particular spectral index (13–16). Stochastic acceleration and shock acceleration are common in many astrophysical settings, from the solar corona to the interstellar medium. Clearly, if we can understand why the spectral index has the unique value of -5 , as observed in the heliosphere, we may be able to apply this understanding far more broadly.

We (4) have pointed out that a spectral index of -5 results in circumstances under which the suprathermal tail behaves as a turbulent cascade. No energy is deposited in the tail, because in this energy range the particles do an equal amount of work on the turbulence as the turbulence does on the particles. Rather, energy flows through the suprathermal tail as in the inertial range of a turbulent cascade (e.g., a Kolomogorov spectrum). This argument has a certain appeal, in part because of its simplicity. However, it does not reveal the underlying physics that demands the unique spectral shape, nor does it place any other constraints on the behavior of the suprathermal tail, which may allow us to extend this result to other astrophysical settings.

In this article we revisit our results (4) and put them on a firmer theoretical foundation. We show that we can apply the

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Abbreviations: SWICS, Solar Wind Ion Composition Spectrometer; AU, astronomical units.

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principles of thermodynamics both to derive the same result that we obtained previously (4) and to explain it. We also derive some additional constraints on the pressure contained in the suprathermal tails and show that these constraints are consistent with observations in the solar wind.

We begin by carefully defining the problem we are considering. We then describe the thermodynamic constraints that must apply in equilibrium. We show how these thermodynamic constraints result in the observed spectral shape and in a formula that constrains the pressure in the suprathermal tail; this formula also is shown to be consistent with observations. We then reconcile these results with our previous cascade arguments (4). In *Concluding Remarks* we discuss how these results can be extended to particle acceleration at shocks.

Defining the Problem

We start with compressional turbulence, which is defined as turbulent motions, with random velocities, $\delta\mathbf{u}$, that result in random compressions and expansions. We thus have random values, equally positive and negative, of $\nabla \cdot \delta\mathbf{u}$.

We consider that present in the turbulence are distinct populations of particles. Consider the solar wind. Solar-wind thermal particles contain the mass and execute the turbulent motions. Present in the solar wind are other, separate populations of particles such as interstellar pickup ions or even a suprathermal tail on the solar-wind thermal distribution. These other populations are affected by, but do not necessarily execute, the turbulent motions. It is the behavior of these latter, separate populations of particles that we are interested in.

The Core Particles. At low particle velocities, a separate population of particles (e.g., the pickup ions) will be alternately compressed and expanded by the compressional turbulence. The particles will behave as do the thermal particles and experience adiabatic compressions and expansions. There is no net energy gain to the separate particle population and no loss to the turbulence. There is no dissipation at this point. We refer to these low-velocity particles as “core particles.”

The Tail Particles. At some higher particle velocity, the separate population particles are sufficiently mobile along the magnetic field so that the particles experience a statistically significant random sample of compressions and expansions. Some particles will experience more compressions than expansions, and vice versa. The particles thus diffuse in velocity or energy, upward toward higher energies, and create a suprathermal tail on the core distribution. This is a classic stochastic acceleration process. We refer to the accelerated particles, in the suprathermal tail, as “tail particles.”

From the perspective of the turbulence, however, it is still only executing adiabatic compressions and expansions on the separate population of particles. There is no dissipation, no transfer of energy from the turbulence to the separate particle population. What the turbulence is doing is redistributing energy from the core to the tail.

The construct of our problem defined here is what places the thermodynamic constraints on the suprathermal tail. The energy that creates the suprathermal tail does not come from the turbulence. Rather, it comes from, and is constrained by, the energy in the core particles. The role of the turbulence is only to redistribute the energy from the core to the tail.

We treat the system of core and tail particles as a thermally isolated system. Thermal isolation, of course, is relative. We are assuming, then, that any escape of particles from the system occurs slowly compared with the time required to establish equilibrium. When we discuss the spectral shape in terms of a cascade in energy, we will, in fact, allow for a slow leakage of the particles out of the system when they reach high energies. Our

assumption, then, is that the leakage occurs on time scales long compared with the establishment of an equilibrium between the core and tail particles.

Thermodynamic Constraints

The stochastic acceleration of the tail particles, which is a diffusion in energy, is an irreversible process that increases the entropy of the system of core and tail particles. No energy is added to this system of core and tail particles; the system is thermally isolated, because there is no dissipation of the turbulence. The creation of the tail particles is analogous to a Joule expansion of a gas into a vacuum, in which the volume of a thermally isolated gas increases irreversibly without performing any work.

The equilibrium state of a thermally isolated system is a state of maximum entropy; that is, in equilibrium the entropy of the tail must attain a maximum. The tail is bounded in energy. At low energy, T_l (or speed v_l), the bound is the interface between the core and the tail (where the particles have sufficient speed to enter into the stochastic acceleration process). There should also be a bound at an upper energy, T_u , where the particle gyroradii exceed the scale size of the turbulence and the tail particles can no longer experience the random compressions and expansions. Thus, equilibrium should occur when the entropy of the tail attains its maximum allowable value within these bounds.

In a thermally isolated system, no process can decrease the entropy and so, after attaining the maximum entropy, the entropy must be constant. The tail particles, however, are still undergoing compressions and expansions by the turbulence. Thus, when the tail particles attain maximum, constant entropy, these expansions and compressions must be isentropic. That is, the pressure in the tail particles, P_t , must undergo reversible, adiabatic compressions and expansions, and thus satisfy

$$\frac{\partial P_t}{\partial t} + \delta\mathbf{u} \cdot \nabla P_t = -\frac{5}{3} (\nabla \cdot \delta\mathbf{u}) P_t, \quad [1]$$

where, again, $\delta\mathbf{u}$ is the random velocity of the turbulent motions. The same situation arises in a Joule expansion. The expansion of the gas is irreversible, but once an equilibrium state is obtained, the state variables of the gas behave as they would under any normal reversible process.

We can also estimate the maximum energy density or, equivalently, the maximum pressure in the tail by considering a compression of the core and the tail. A compression of the core particles will increase their density and temperature. The increase in temperature will raise the speed of some of the core particles to be above the threshold speed of the tail. The core particles that cross this threshold speed can thus flow into the tail. When the energy density contained in the tail equals the energy density of the core that rises above the threshold energy, T_l , no flow of energy from the core to the tail should be possible, and the entropy of the tail remains unchanged.

When the core particles are compressed, the increase in pressure associated with the increase in temperature, δP_c , can readily be shown to be

$$\delta P'_c = \frac{\gamma - 1}{\gamma} \delta P_c. \quad [2]$$

Here, δP_c is the total increase in pressure of the core particles, and γ is the ratio of specific heats. For our problem, $\gamma = 5/3$, or $(\gamma - 1)/\gamma = 2/5$.

An equilibrium occurs when the energy that can flow from the core to the tail, Eq. 2, is equal to the average pressure in the tail particles, P_t , plus the pressure in the compression of the tail particles, δP_t . The compression of the core, which can result in the flow of energy from the core to the tail, will also result in a

compression of the tail particles. In equilibrium, then, where there is no flow of energy from the core to the tail, we expect that

$$\frac{2}{5} \delta P_c = P_t + \delta P_t. \quad [3]$$

We have, then, two main thermodynamic constraints on our system.

1. The system of core and tail particles is thermally isolated, and in equilibrium the tail contains the maximum allowed entropy. Any subsequent compression or expansion of the pressure of the tail must be isentropic and satisfy Eq. 1.
2. The pressure in the tail and its fluctuations is limited by the pressure in the core particles, according to Eq. 3.

From these two constraints the unique spectral shape will follow, as well as our previous arguments (4).

The Unique Spectral Shape

The spectral shape and the behavior of the suprathermal tail can best be understood by expressing the spectrum of the tail particles in terms of the differential number density, U_t , the number of particles per unit volume and per unit kinetic energy, T . A distribution function with a spectral index of -5 (i.e., $f_t \propto v^{-5}$, where v is particle speed) corresponds to a differential number density spectrum, $U_t \propto T^{-2}$ (i.e., the required spectral index is -2).

The equation that governs the behavior of the tail particles is the standard transport equation for energetic particles, first derived by Parker (17) and Gleeson and Axford (18),

$$\frac{\partial U_t}{\partial t} + \nabla \cdot (\mathbf{u} U_t) = \nabla \cdot (\kappa \nabla U_t) + (\nabla \cdot \mathbf{u}) \frac{\partial}{\partial T} \left(\frac{2}{3} T U_t \right). \quad [4]$$

Here, $\mathbf{u}$ is the convective flow velocity, and κ is the spatial diffusion tensor.

We transform into the frame moving with the mean flow velocity. Thus, $\mathbf{u} = \delta \mathbf{u}$, the random flow velocity. In the case of the solar wind, this transformation will neglect adiabatic deceleration in the expanding solar wind, under the assumption that the stochastic acceleration is a much faster process. We also neglect diffusive transport. The tail is part of an isolated system, and escape by diffusion is not permitted. For example, the particles are mobile along the magnetic field, and with homogeneous conditions no gradients or net transport parallel to the magnetic field can be maintained. There will be gradients normal to the magnetic field, but we assume that there is no cross-field diffusion. Eq. 4 then becomes

$$\frac{\partial U_t}{\partial t} + \delta \mathbf{u} \cdot \nabla U_t = -(\nabla \cdot \delta \mathbf{u}) \left[U_t - \frac{\partial}{\partial T} \left(\frac{2}{3} T U_t \right) \right]. \quad [5]$$

We expect that stochastic acceleration in compressional turbulence will yield a power-law solution in kinetic energy for U_t of the form

$$U_t = g(\mathbf{r}, t) T^{-\alpha}. \quad [6]$$

Such a solution, with $\alpha = 2$, is required to be consistent with observations. Moreover, such a solution is expected, because the mean rate of change in energy in an expansion or a compression is $\Delta T / \Delta t \propto T$, a form that supports power-law solutions. Thus, with the form of the solution in Eq. 6, Eq. 5 reduces to

$$\frac{\partial U_t}{\partial t} + \delta \mathbf{u} \cdot \nabla U_t = - \left(1 + \frac{2}{3} (\alpha - 1) \right) (\nabla \cdot \delta \mathbf{u}) U_t. \quad [7]$$

We also argued that the tail is bounded in energy. At low energies, T_l , the bound is the interface between the core and the tail; at high energies, T_u , the bound is where the particle gyroradii exceed the scale size of the turbulence and the tail particles can no longer experience the random compressions and expansions. The pressure of the tail particles is then

$$P_t = \int_{T_l}^{T_u} \frac{2}{3} T U_t dT. \quad [8]$$

Finally, we convert Eq. 7 into an equation for the pressure, P_t , by multiplying through by $2T/3$ and integrating from T_l to T_u , or

$$\frac{\partial P_t}{\partial t} + \delta \mathbf{u} \cdot \nabla P_t = - \left(1 + \frac{2}{3} (\alpha - 1) \right) (\nabla \cdot \delta \mathbf{u}) P_t. \quad [9]$$

The thermodynamic constraint is that P_t satisfies Eq. 1. Thus, comparing Eqs. 9 and 1, we conclude that α must be 2. A power law in energy with a spectral index of -2 is unique. It is the one spectrum in which the pressure in the tail, by itself, undergoes isentropic compressions and expansions. When the stochastic acceleration has maximized the entropy of the tail, subsequent compressions and expansions must be isentropic.

The spectral shape of the tail can also be understood as arising from a flow of energy through the tail, as in a turbulence cascade (4), an argument that is developed further below.

Constraints on the Pressure of the Tail Particles

In equilibrium, the pressure of the core and tail, thus, must each behave according to Eq. 1, for an isentropic compression or expansion. Thus,

$$\frac{1}{P_t} \left[\frac{\partial P_t}{\partial t} + \delta \mathbf{u} \cdot \nabla P_t \right] = -\frac{5}{3} (\nabla \cdot \delta \mathbf{u}) = \frac{1}{P_c} \left[\frac{\partial P_c}{\partial t} + \delta \mathbf{u} \cdot \nabla P_c \right], \quad [10]$$

from which it readily follows that the relative fluctuations in the core and tail pressure must be equal, or

$$\frac{\delta P_c}{P_c} = \frac{\delta P_t}{P_t} \equiv \beta. \quad [11]$$

Thus, Eq. 3 can be written as

$$P_t = \frac{2}{5(\beta + 1)} P_c, \quad [12]$$

where β should be interpreted as the maximum relative increase in pressure of the tail or core; this value of β yields the maximum energy that can flow from the core into the tail and its fluctuations. We do not expect that β should ever exceed unity, from which we conclude that the tail pressure should never exceed 20% of the core pressure.

With our thermodynamic constraints, we thus conclude that the pressure in the tail is proportional to the pressure in the core, with the proportionality constant depending on the fluctuations in the pressure. These quantities are all observable; thus, the consistency of these results with observations can be tested.

Consistency of the Results with Observations

We can estimate the relative fluctuations of the core particle pressure from the observed counting rates of this population. In Fig. 2 we show the 2-hour averaged counting rates for 100-day periods of core protons in the high-latitude, least-disturbed, high-speed solar wind in the polar coronal hole during solar minimum at ≈ 3 astronomical units (AU) from days 100 to 200 in 1996 (*Top*); of core protons during the least-disturbed or

turbulent speed of the fluctuations with wavenumber k . The wavenumber in the inertial range changes in a characteristic time $\tau_k \sim (k\delta u_k)^{-1}$. The rate of flow of energy through the inertial range is then

$$\varepsilon \sim \frac{k}{\tau_k} E_k = k^{5/2} E_k^{3/2}. \quad [19]$$

The requirement that ε is constant, independent of k in the inertial range, yields the standard Kolomogorov result that $E_k \propto k^{-5/3}$ (19).

In the case of the suprathermal tail, the differential energy density is $E_t = TU_t$. The energy changes at a rate $\Delta T/\Delta t \propto T$ [e.g., the work done by the turbulence on the tail particles is $\delta u \cdot \nabla(2TU_t/3)$ in Eq. 15]. Thus, the rate of flow of energy through the tail is

$$\varepsilon \sim \frac{\Delta T}{\Delta t} TU_t \propto T^2 U_t. \quad [20]$$

The requirement that ε is constant, independent of T , thus requires that $U_t \propto T^{-2}$, the required spectral shape for the suprathermal tails.

Concluding Remarks

We have concluded that the unique spectral shape observed for suprathermal tails in the solar wind can be understood by applying thermodynamic constraints to stochastic acceleration in compressional turbulence. The system of core and tail particles is thermally isolated, and thus the stochastic acceleration will result in the entropy of the tail particles increasing to a maximum, constant value. It follows, then, that the compressions and expansions of the pressure of the tail particles must be isentropic, which for the observed power-law solutions requires a unique spectral shape. When expressed as a distribution function, the spectral index must be -5 ; when expressed in terms of differential number density, the spectral index is -2 ; and when expressed in terms of differential intensity, the spectral index is -1.5 .

This result, as well as the result limiting the pressure in the suprathermal tail, is quite general. We require only stochastic acceleration by compressional turbulence, in a thermally isolated system. We thus require that the time scale for the stochastic acceleration is small compared with any escape of the particles from the system (e.g., caused by spatial diffusion or escape at high energies when the particle gyroradii exceed the scale size of the turbulence).

We also require, of course, that the stochastic acceleration in compressional turbulence invoked here is the most rapid acceleration process in the plasma, which is generally not difficult to achieve. Stochastic acceleration in compressional turbulence is similar to transit-time damping of magnetic field magnitude fluctuations (e.g., compressions/expansions of the magnetic

field), which Fisk (8, 20) showed can be much faster than other forms of acceleration in the solar wind (e.g., second-order Fermi acceleration caused by pitch-angle scattering, or adiabatic deceleration in the expanding solar wind).

Within these general constraints and assumptions, the results presented in this article should apply, for example, in coronal loops at the Sun, in the solar corona, or perhaps in more general astrophysical settings. It should be possible, then, in these settings to both specify the likely spectral shape of the accelerated particles and relate their intensity to other aspects of the plasma, such as the pressure of the core particles and their fluctuations.

One potentially interesting extension of this work is to the acceleration of energetic particles at shocks. The redistribution of energy by stochastic acceleration, from the core to the tail, could occur both upstream and downstream from the shock. If this is the case, we would conclude from Eq. 12 that the core and tail pressure are proportional. We can then apply the standard Rankine-Hugoniot relationships for the pressure jump across the shock and, in particular, find the jump in the tail pressure. This approach was applied by Fisk *et al.* (6) to the termination shock of the solar wind. Acceleration of energetic particles at shocks is likely to be more complicated. The core particles are heated nonadiabatically when crossing a shock, which raises more of them above the threshold for injection into the tail than is predicted in Eq. 3, and this will yield an enhanced suprathermal tail downstream. Such an enhanced tail is consistent with our result that in disturbed downstream conditions, the simple pressure equilibrium formula (Eq. 12) does not hold. The application of stochastic acceleration in compressional turbulence to shocks needs to be examined in more detail and compared with observations of the many shocks available in the solar wind.

Finally, we need to comment on why the conclusions of this article are not evident in the many calculations that have been performed on stochastic acceleration. In general, such calculations take a test particle or quasi-linear approach and do not yield a unique spectral shape. Such calculations should be applicable to the evolution of the spectrum with time. However, they do not include the thermodynamic constraints that must apply when an equilibrium is achieved: the energy in the tail is limited by the energy in the core particles, not the turbulence; in equilibrium, the pressure in the tail must be compressed and expanded isentropically. It is these thermodynamic constraints that yield the unique spectral shape and must be included, because the constraints are simply the first and second laws of thermodynamics.

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