Oscillations of plants' stems and their damping: theory and experimentation

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Free oscillations of upright plants' stems, or in technical terms slender tapered rods with one end free, can be modelled by considering the equilibrium between bending moments and moments resulting from inertia. For stems with apical loads and negligible mass of the stem and for stems with finite mass but without top loading, analytical solutions of the differential equations with appropriate boundary conditions are available for a finite number of cases. For other cases approximations leading to an upper and a lower estimate of the frequency of oscillation ω can be derived. For the limiting case of $\omega = 0$, the differential equations are identical with Greenhill's equations for the stability against Euler buckling of slender poles. To illustrate, the oscillation frequencies of 25 spruce trees (*Picea sitchensis* (Bong.) Carr.) were compared with those calculated on the basis of their morphology, their density and their static elasticity modulus. For *Arundo donax* L. and *Cyperus alternifolius* L. the observed oscillation frequency was used in turn to calculate the dynamic elasticity modulus, which was compared with that determined in three-point bending. Oscillation damping was observed for *A. donax* and *C. alternifolius* for plants' stems with and without leaves or inflorescence. In *C. alternifolius* the difference can be attributed to the aerodynamic resistance of the leaves, whereas in *A. donax* structural damping in addition plays a major role.

Keywords: plant stems; oscillations; aerodynamics; structural damping; differential equations

1. INTRODUCTION

Plants oscillate when reacting to stimuli such as gusts of wind. Knowledge of the resonance frequency, particularly in the fundamental mode of oscillation, is necessary to assess a plant's stability against dynamic wind loads (Kerzenmacher & Gardiner 1998). In addition, measurement of the frequency of oscillations of plants' stems or slender rods offers an efficient and potentially non-destructive way to determine mechanical properties such as storage and loss moduli of elasticity (Young 1989; Vincent 1990; Zebrowski 1991; Niklas 1992; Speck & Spatz 2000). Spatz & Speck (2002) described oscillations in the vertical and horizontal orientations of plants' stems with or without apical loads, with different tapering modes and with changes in the elasticity modulus and the density from the base to the apex.

Dynamic wind loads can excite potentially destructive swaying in plants. Kerzenmacher & Gardiner (1998) have shown that the energy transfer is most effective in the fundamental oscillation mode. The dissipation of mechanical energy is the prerequisite of oscillation damping and thus a means of escaping a 'resonance catastrophe'. Although strict separation may not be possible, oscillation damping in isolated plants' stems can be attributed to structural damping, aerodynamic damping and damping as a property of the material comprising the stem (Niklas 1992).

Structural damping (Kerzenmacher & Gardiner 1998) is observed in plants that directly after release do not oscillate uniformly. Depending on the mass distribution and stiffness of the plant's organs, leaves or inflorescences or even the upper parts of the plant oscillate with a different frequency or phase from that of the main parts of the stem. Inertial forces and aerodynamic resistance induce this complex behaviour. As shown for *Arundo donax* (Speck & Spatz 2003) structural damping can be equally or even more effective than the actual aerodynamic damping, especially in the initial phases of the oscillation.

This paper gives an outline of the solutions of the differential equations for bending vibrations in the fundamental mode and delineates their approximations. Preference is given to analytical solutions because they can be more easily generalized. The applicability for plants' stems is exemplified by quantitative recordings of oscillations of spruce trees, and the grasses *Cyperus alternifolius* and *A. donax*.

2. THEORETICAL CONSIDERATIONS

(a) *Top loading with negligible stem mass*

The differential equation for free vibrations of an upright slender rod or stem (figure 1) with an apical load can be formulated as an equilibrium of moments. The first term describes the moment due to bending of the stem. The second term is the moment induced by the weight of the top load. The third term results from the inertia of the masses accelerated. Plants' tissues, particularly strengthening tissues, obey Hooke's law at small strains.

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Figure 1. Scheme to illustrate a slender tapered plant's stem with an apical load, oscillating around an upright resting position.

Neglecting the mass of the stem itself we obtain the following for deflections within the linear elastic range:

$$EI \frac{d^2 y}{dx^2} + Mg(y - y_T) - (\omega^2 + \delta^2)M(x - x_T)y_T = 0, \quad (2.1)$$

with x the coordinate along the length of the stem ($x = L$ at the base and $x = 0$ at the tip of the stem); x_T is the position of the top load; $y(x)$ is the deflection from the resting position; y_T is the deflection at $x = x_T$; L is the length of the stem; $E(x)$ is the elasticity modulus; $I(x)$ is the axial second moment of area; M is the mass of the apical load; g is the gravitational acceleration; ω is the angular frequency; and δ is the decay constant. For heterogeneous material $E(x)$ is a 'structural modulus of elasticity', a weighted average over the different tissues and their distribution in the cross section.

For plants' stems it is reasonable to assume that EI varies smoothly and monotonously over its length. We can approximate

$$EI = E_B I_B z^{4\alpha + \beta}, \quad (2.2)$$

where $z = x/L$ and correspondingly $dx = Ldz$; E_B is the elasticity modulus at the base; I_B is the axial second moment of area at the base; α is the tapering mode

($r = r_B z^\alpha$ or $I = I_B z^{4\alpha}$); and β is the mode of dependence of the elasticity modulus ($E = E_B z^\beta$).

With $\psi = y/y_T$ the differential equation (2.1) can be rewritten in the form

$$\frac{d}{dz} \left(z^{4\alpha + \beta} \frac{d^2 \psi}{dz^2} \right) + B \frac{d\psi}{dz} - A = 0, \quad (2.3a)$$

with

$$B = L^2 \frac{Mg}{E_B I_B} \quad (\text{the 'gravitational term'}) \quad (2.3b)$$

and

$$A = L^3 \frac{M}{E_B I_B} (\omega^2 + \delta^2) \quad (\text{the 'acceleration term'}). \quad (2.3c)$$

The boundary conditions read

$$\psi(1) = 0, \quad \frac{d\psi}{dz}(1) = 0 \quad \text{for the base} \quad (2.3d)$$

and

$$\frac{d^2 \psi}{dz^2}(z_T) = 0 \quad (2.3e)$$

for the relative position of the top load.

As shown by Spatz & Speck (2002) the case of a horizontal orientation of the stem is a special case of equations (2.3) with $B = 0$. For $A = 0$ (i.e. $\omega = 0$) equations (2.3) become identical to Greenhill's equation for the stability of a top-loaded slender rod against Euler buckling (Greenhill 1881; Spatz 2000).

The remaining problem is to find numerical values for combinations of A and B , which characterize the fundamental frequency of free vibration. The equivalent condition for any given value of B is the solution of differential equation (2.3a) with the boundary conditions in equation (2.3d), which fulfils $\psi(z_T) = 1$ (Spatz & Speck 2002).

(b) Stems without top loading

The differential equation for free vibrations of an upright slender rod or stem without an apical load can be formulated as an equilibrium of line loading.

$$\frac{d^2}{dx^2} \left(EI \frac{d^2 y}{dx^2} \right) + \frac{d}{dx} \left\{ g \left[\int_0^x \rho a d\xi \right] \frac{dy}{dx} \right\} - (\omega^2 + \delta^2) \rho a y = 0, \quad (2.4)$$

where a is the cross-sectional area with $a = a_B z^{2\alpha}$; a_B is the cross-sectional area at the base; and $\rho(x)$ is the density of the stem at a particular height. For plants' stems the variation of ρ along the stem can be approximated by $\rho = \rho_B z^\gamma$ with $z = x/L$, where ρ_B is the density at the base and γ the mode of dependence.

Equation (2.4) can be rewritten in the form

$$\frac{d^2}{dz^2} \left(z^{4\alpha + \beta} \frac{d^2 y}{dz^2} \right) + G \frac{d}{dz} \left(\frac{z^{2\alpha + \gamma + 1}}{2\alpha + \gamma + 1} \frac{dy}{dz} \right) - H z^{2\alpha + \gamma} y = 0, \quad (2.5a)$$

with

$$G = L^3 \frac{\rho_B a_B g}{E_B I_B} \quad (\text{the 'gravitational term'}) \quad (2.5b)$$

and

$$H = L^4 \frac{\rho_B a_B}{E_B I_B} (\omega^2 + \delta^2) \quad (\text{the 'acceleration term'}). \quad (2.5c)$$

The boundary conditions read

$$y(1) = 0 \quad \frac{dy}{dz}(1) = 0 \quad \text{for the base} \quad (2.5d)$$

and

$$\frac{d^2 y}{dz^2}(0) = 0 \quad \frac{d^3 y}{dz^3}(0) = 0 \quad \text{for the tip.} \quad (2.5e)$$

Horizontal orientation of the stem is a special case of equations (2.5) with $G = 0$. For $H = 0$ (i.e. $\omega = 0$) equations (2.5) become identical to Greenhill's equation for the stability of a slender pole without top loading (Greenhill 1881; Spatz 2000).

The remaining problem is to find numerical values for H and G , which characterize the fundamental frequency of free vibration. The equivalent condition is the least positive singularity in the solution of the differential equation with the above boundary conditions. MATHEMATICA v. 4.0 solves differential equations (2.5) only for $G = 0$ or $H = 0$ (Spatz & Speck 2002). Approximations for G and $H \neq 0$ can be obtained as follows.

(c) Approximations

(i) Top loading with negligible stem mass

The differential equations (2.3) are only solved for $4\alpha + \beta = 0, 2$ or 4 with independent α and β values. For other values of α and β solutions are only provided for $A = 0$ or $B = 0$. For all other cases approximations can be obtained by considering the potential energy during oscillations (Timoshenko & Young 1948).

The strain energy of bending the stem is

$$V_1 = \frac{1}{2} \int_{x_T}^L \text{moment curvature } dx.$$

Within the linear elastic range where moment = EI curvature and with $z = x/L$ and equation (2.2) this reads

$$V_1 = \frac{1}{2} \frac{E_B I_B}{L^3} \int_{z_T}^1 z^{4\alpha+\beta} \left(\frac{d^2 y}{dz^2} \right)^2 dz. \quad (2.6)$$

During the oscillation the top load experiences a vertical displacement Δh . The corresponding potential energy is

$$V_2 = -Mg\Delta h \approx -\frac{1}{2} \frac{Mg}{L} \int_{z_T}^1 \left(\frac{dy}{dz} \right)^2 dz. \quad (2.7)$$

Because the oscillation converts the potential energy (V) to kinetic energy (T) and vice versa, the balance for an undamped oscillation reads

$$V = V_1 + V_2 = T = \frac{1}{2} M \omega^2 y_T^2. \quad (2.8)$$

With $\psi = y/y_T$, equations (2.3a,b) and the abbreviations

$$K_1 = \int_{z_T}^1 z^{4\alpha+\beta} \left(\frac{d^2 \psi}{dz^2} \right)^2 dz \quad \text{and} \quad K_2 = \int_{z_T}^1 \left(\frac{d\psi}{dz} \right)^2 dz,$$

lead to the linear relation

Figure 2. The acceleration term, A , plotted against the gravitational term, B , for $\alpha = 1$, $\beta = 0$ and $z_T = 0.05$. The squares represent the solutions of differential equations (2.3). The figure illustrates the procedure as described in § 2c. The straight line through A_0 and B_0 gives a lower estimate, the lines from A_0 or B_0 to the intercept of the two straight lines defined by equation (2.9) provide an upper estimate. $A^* = K_1$ for $B \approx 0$, $B^* = K_1/K_2$ for $A \approx 0$. The approximations improve as α becomes smaller.

$$A = K_1 - K_2 B = A_0 - K_2 B. \quad (2.9)$$

The evaluation of the numerical terms K_1 and K_2 requires knowledge of the bending line $\psi(z)$. Timoshenko & Young (1948) used a cosine function. A better approximation, especially for strongly tapered columns, uses the bending lines for the two limiting cases: for $A \approx 0$, $\psi(z)$ is chosen as the solution of equations (2.3) with $A = 0$, i.e. Greenhill's equation for Euler buckling. For $B \approx 0$, $\psi(z)$ is chosen as the solution of differential equations (2.3) with $B = 0$, i.e. zero gravity or horizontal orientation, provided that the downward deflection in its resting position is small.

This is illustrated for $\alpha = 1$ in figure 2. Applying equation (2.7) and choosing a solution of equations (2.3) for $A = 0$ leads to a straight line through $(0, A_0)$ and $(B^*, 0)$, choosing a solution of equations (2.3) for $B = 0$ leads to a straight line through $(0, A^*)$ and $(B_0, 0)$, where $A_0 = A(B = 0)$ and $B_0 = B(A = 0)$ are exact solutions. The lines from $(0, A_0)$ or $(B_0, 0)$ to the intercept of the two straight lines represent an upper estimate of the actual data points, a straight line through $(0, A_0)$ and $(B_0, 0)$ a lower estimate.

(ii) Stems without top loading

The differential equations (2.5) are only solved by MATHEMATICA v. 4.0 for $G = 0$ (zero gravity) or $H = 0$

(Greenhill's equation). For all other cases approximations have to be derived by a procedure similar to that for stems with top loading.

The strain energy of bending the stem can be calculated analogously to equation (2.6):

$$V_1 = \frac{1}{2} \frac{E_B I_B}{L^3} \int_0^1 z^{4\alpha+\beta} \left(\frac{d^2 y}{dz^2} \right)^2 dz. \quad (2.10)$$

During the oscillation each part of the stem experiences a vertical displacement. The corresponding potential energy is

$$V_2 = -\frac{1}{2} g \rho_B a_B \int_0^1 z^{2\alpha+\gamma} \left[\int_1^z \left(\frac{dy}{d\xi} \right)^2 d\xi \right] dz. \quad (2.11)$$

The kinetic energy can be calculated as

$$T = \frac{1}{2} \rho_B a_B L \omega^2 \int_0^1 z^{2\alpha+\gamma} y^2 dz. \quad (2.12)$$

For an undamped oscillation, the energy balance $V = V_1 + V_2 = T$ with equations (2.5b,c) and the abbreviations

$$M_1 = \int_0^1 z^{4\alpha+\beta} \left(\frac{d^2 y}{dz^2} \right)^2 dz$$

$$M_3 = \int_0^1 z^{2\alpha+\gamma} \left[\int_1^z \left(\frac{dy}{d\xi} \right)^2 d\xi \right] dz$$

$$M_4 = \int_0^1 z^{2\alpha+\gamma} y^2 dz$$

leads to

$$H = \frac{M_1}{M_4} - \frac{M_3}{M_4} G = H_0 - \frac{M_3}{M_4} G. \quad (2.13)$$

The bending line $y(z)$ for $H \approx 0$ is approximated as the solution of differential equations (2.5) for $H=0$, i.e. Greenhill's equation. The bending line $y(z)$ for $G \approx 0$ is approximated as the solution of differential equations (2.5) for $G=0$, i.e. zero gravity.

These energy considerations (Timoshenko & Young 1948) can even be generalized to treat the case of upright tapered stems with finite mass and an additional load (mass) attached at one point along the stem (Spatz & Speck 2002). For most applications these approximations will suffice. Numerical search for solutions of the differential equations may lead to even better approximations. This would have the additional advantage that more complex objects could be included in the analysis.

3. APPLICATIONS

Thirty-three spruce trees (*Picea sitchensis*) from within a stand at Kilmichael Forest, Scotland, UK, were investigated. The bending stiffness EI along the stem was measured in pulling experiments with eight extensometers as described previously (Brüchert *et al.* 2000; Brüchert & Gardiner 2000). Swaying was initiated by a sudden release of the pull and recorded as the output of the extensometers. The morphology of each tree, the mass of the branches and the density of the stem along its length was determined after felling (Brüchert & Gardiner 2000).

Figure 3. Comparison of the swaying frequencies observed for 25 Sitka spruce trees and the frequencies calculated according to equations (2.5) and (2.13).

To compare the calculated oscillation frequencies with those observed, the tapering mode α , the variation of the bending stiffness along the stem $4\alpha + \beta$, and the variation of the mass along its length $\gamma_{\text{effective}}$ have to be known. The effective density is obtained by adding the mass of the branches to the mass of the stem at every section. Not all trees conformed to a variation of their properties in the form of a power function. Only 25 trees, where a straight line in a double logarithmic plot correlated with the data points with $R^2 > 0.95$ for the determination of α , $R^2 > 0.95$ for $4\alpha + \beta$ and $R^2 > 0.75$ for $\gamma_{\text{effective}}$, were included in the analysis.

In plants a gradual transition of the mechanical properties from stem to roots is typically realized by a decrease in the elasticity modulus near the base and a corresponding increase in the radius. Therefore extrapolated values were used for the cross-sectional area a_B and the bending stiffness $E_B I_B$ at the base. Figure 3 compares the frequencies observed to those calculated by equations (2.5) for upright stems without top loading. The means differ by only 7%, nevertheless the individual prediction can be as inaccurate as 30%.

Good agreement of the calculations with the experiments was also found for isolated stems of *A. donax* and *C. alternifolius*. In this comparison the frequencies recorded were used to calculate the dynamic elasticity modulus, which was compared to that from three-point-bending experiments. For *A. donax* a difference of less than 10% was found (Spatz & Speck 2002). For *C. alternifolius* the values matched within $\pm 20\%$.

Frame by frame analyses of video recordings (figure 4) show that the swaying of *A. donax* can be described as a damped harmonic bending oscillation (Speck & Spatz 2000). Damping can be characterized by logarithmic decrement or equivalent by the relative damping E''/E' , where E'' is the loss modulus, E' is the storage modulus and E is the absolute value of the complex elasticity modulus (Vincent 1990).

The relative damping is a measure of the dissipation of mechanical energy which can occur by

- (i) friction with other members of the plant community as, for instance, in a dense stand;
- (ii) structural damping (Niklas 1992; Kerzenmacher &

Figure 4. Displacement at various heights above ground as function of time for an *Arundo donax* without leaves. The lines through the data points correspond to an equation for a damped harmonic oscillation.

Gardiner 1998), i.e. by oscillation of parts of the plant with different frequencies or phases;

- (iii) aerodynamic damping; and
- (iv) damping due to the viscoelastic properties of the material comprising the plant's stem or branch, which is bent during oscillations.

This last aspect will be discussed first. Viscoelasticity is a property often, if not always, found in fresh plant material (Vincent 1990; Niklas 1992; Spatz *et al.* 1999; Köhler & Spatz 2002). From a comparative approach a consistent picture emerges that oscillation damping is particularly effective if stiff components such as sclerenchyma and vascular bundles are embedded in a more compliant surrounding like parenchyma to allow for a maximal interface. Upon bending this will lead to shearing and dissipation of mechanical energy.

Damping is much more effective if leaves or inflorescences are present. Using $c_D = 1$ an estimate of the aerodynamic resistance of the leaves of *C. alternifolius* leads to $E''/E' = 0.078 \pm 0.009$ compared with the experimentally determined value of 0.073 ± 0.018 . Damping of the plants' stems that had their leaves replaced by a point mass of plasticine (Spatz & Zebrowski 2001) yields $E''/E' = 0.032 \pm 0.012$. In *C. alternifolius* the difference in damping with and without leaves can therefore simply be explained by aerodynamic damping.

The situation is, however, different for *A. donax* (Speck & Spatz 2003), where aerodynamics can only account for a minor part of damping of an oscillating stem with leaves or inflorescences. The major part must therefore be attributed to structural damping. The phenomenon can be understood by considering damping in a system of coupled oscillators: the aerial parts of a plant should be considered as a system, where owing to material

or aerodynamic damping the frequency band of each member is broad enough to allow energy transfer to others. As the surface to mass ratio is larger in the periphery damping is expected to be most effective in the leaves or the terminal organs. Thus, an ever smaller part of the energy is available to the system. In this sense the capacity to damp oscillations is both an overall as well as a local feature of the plant.

In plants, structural damping is probably induced by point loading before release compared with line loading during oscillations, as well as by the inertia of the leaves or the inflorescence. During the initial phase, the velocity of the movement is quite high, therefore aerodynamic resistance will also induce relative movements of leaves or inflorescence with respect to the stem. As a result, the terminal organs of the plant will oscillate initially with a different frequency or phase than the major part of the stem until later at smaller amplitudes they will be brought in line. Thus, structural damping is expected to play a larger role during the first periods of oscillation than for the later ones.

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