Red spectra from white and blue noise

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The value of maps of the interval in modelling population dynamics has recently been called into question because temporal variations from such maps have blue or white power spectra, whereas many observations of real populations show time-series with red spectra. One way to deal with this discrepancy is to introduce chaotic or stochastic fluctuations in the parameters of the map. This leads to on-off intermittency and can markedly redden the spectrum produced by a model that does not by itself have a red spectrum. The parameter fluctuations need not themselves have a red spectrum in order to achieve this effect. Because the power spectrum is not invariant under a change of variable, another way to redden the spectrum is by a suitable transformation of the variables used. The question this poses is whether spectra are the best means of characterizing a fluctuating variable.

Keywords: dynamical systems; on-off intermittency; population dynamics; environmental effects; $1/f$ spectra

1. INTRODUCTION

One of the first applications of maps of the interval to the natural sciences was in the study of the time variations of populations of groups of organisms. Such maps are often chaotic and it then becomes difficult to make detailed comparisons between theory and experiment (Diamond & May 1977; Pimm & Redfearn 1988). One way of characterizing the data for this purpose is to compute the power spectrum from the time-series of population fluctuations. Such spectra have been observed in real populations to be distinctly red, with more power in their low frequency variations than in the high frequencies (Sugihara 1995; Steele 1985). By contrast, spectra formed from theoretical time-series derived from models of populations of single species with non-overlapping generations have been reported to be blue, or sometimes white, but certainly not red (Cohen 1995). This discrepancy, referring as it does to a much-used statistical descriptor of time-series, may call for some further understanding and modification of the models if they are to account for the observed tendencies of population dynamics (Sugihara 1995; Blarer & Doebeli 1996; White *et al.* 1996*a,b*; Kaitala & Ranta 1996; Kaitala *et al.* 1997; Miramontes & Rohani 1998). The issues raised could have implications for other disciplines that use simple chaotic models.

2. MECHANISMS OF SPECTRAL REDDENING

In this work, we discuss how parameter fluctuations with blue or white spectra can induce reddening of the spectra from chaotic models that would not by themselves lead to red spectra. But before turning to such effects, we observe that red spectra may result purely from slow evolution of

the mean properties of the population: a slow increase of the running average of the population can cause reddening of the power spectrum of a time-series. A good example is provided by data on variations in locust populations taken over a span of 1000 years (Ma 1958). The data are seen to show a slow drift in mean population over the observation period in which the spectra are computed (Sugihara 1995), with consequent reddening. Secular drifts of this kind may represent real effects and have been discussed, for instance, in climatology (Allen & Smith 1994). However, we are concerned with situations in which mean drifts have been suppressed, as is routinely done for data sets in economics, for example (Brockwell & Davis 1987).

It has also been reported (Blarer & Doebeli 1996; White *et al.* 1996*b*) that not all unimodal maps have the properties suggested by the results of Cohen (1995). Even a map that produces a blue spectrum may, under an analytic change of variables, be turned into a map that can generate a red spectrum. To illustrate how simple transformations can generate maps with red spectra from maps with blue ones we consider the logistic map,

$$x(n+1) = Rx(n)[1 - x(n)], \quad (1)$$

which is a reparameterized version of the Verhulst model (May & Oster 1976) used by Cohen (1995). Here n is the (discrete) time variable, $x(n)$ is the population at time n and R is the control parameter characterizing the environment.

Figures 1–3 respectively, show the map, a time sequence it produces and the average power spectrum for $R = 4$. Also shown are the corresponding pictures for the transformed variables $y(n) = 1 - e^{-4x(n)}$ and $z(n) = e^{4x(n)}$. Though the spectrum of fluctuations from the logistic map with $R = 4$ is white, transformation to the variable y

Figure 1. The panels show (a) the logistic map and (b) its relatives obtained by the change of variable $y = 1 - e^{-4x}$ and (c) $z = e^{4x}$.

changes it to red, while transformation to z produces a blue spectrum. This lack of invariance of the spectral form under analytic change of variables should cause concern that the power spectrum of a time-series is not a robust diagnostic tool unless a suitable choice of variables may somehow be distinguished.

Other mechanisms reported to produce reddened spectra are time delays (Kaitala & Ranta 1996), the interaction of several maps (White *et al.* 1996a), and stochastic fluctuations in parameters of the map (Sugihara 1995; Kaitala *et al.* 1997; Miramontes & Rohani 1998). In the next section we show that reddening may result also when the environmental variations (fluctuations in map parameters) are described by low-order deterministic systems, as is foreshadowed by the work of Steele & Henderson (1992) on the spatial patchiness in plankton.

3. REDDENING BY ON-OFF INTERMITTENCY

To describe how to redden the spectra from a model by fluctuations in its parameter(s), we suppose that we have already adopted a map, of the form

$$x(n+1) = f(x(n), R(n)), \quad (2)$$

where the control parameter R , reflecting (for example) prevailing environmental conditions, varies with time, n . We restrict attention to the plausible case in which the variations of R are independent of the population under consideration, and write:

$$R(n+1) = g(R(n)). \quad (3)$$

When g is either a deterministic chaotic map or a stochastic process, for wide ranges of parameters, these equations produce what we call 'on-off intermittency' in $x(n)$, in which periods of very low activity are punctuated by bursts of intense and often unpredictable behaviour (Platt *et al.* 1993a; von Hardenberg *et al.* 1997). The process of on-off intermittency has been proposed as a general mechanism for bursting in physical systems (Platt *et al.* 1993b; Ding & Yang 1997; Duane 1997; Landa *et al.* 1998). Some of the quantitative characteristics of on-off intermittent time-series have been discussed, for example, by Ding & Yang (1997) and von Hardenberg *et al.* (1997).

For the system (3)–(4), the on-off character of $x(n)$ arises because for fixed R less than some critical value, R_0 , the fixed point $x = 0$ of equation (2) is stable and the system is 'off'. But when $R > R_0$, that fixed point is unstable, we observe non-zero values of x and the system is 'on'. Now, provided the dynamics of equation (3) are suitably chosen, R varies irregularly over values spanning the threshold. But during certain intervals it is, on average, below R_0 , and over other periods its mean value will be above that threshold. Thus the overall effect will be to switch $x(n)$ on and off. This is the essence of on-off intermittency and the concept may readily be extended to more complicated systems and to higher dimensions.

One of the important features of on-off intermittency is that it generates signals with arbitrarily long time-scales of variability. Hence the time-series of the population, $x(n)$, has a red spectrum. Moreover, this red spectrum results independently of the details of the parameter variations, $R(n)$. Thus, the environmental fluctuations need not themselves have red spectra, as noted for stochastic parameter variations by Kaitala *et al.* (1997).

We again illustrate our point with the logistic map. For the parameter variation we take

$$x(n+1) = [aR(n) + b]x(n)[1 - x(n)], \quad (4)$$

with $R(n)$ given by equation (3). If the variation of R , given in terms of g , is chosen to be white noise equidistributed on the interval $[0, 1]$ with $a = 3.5$ and $b = 0$, we obtain the results in figure 4. The figure shows: (a) the power spectrum of $R(n)$; (b) the power spectrum of $x(n)$; and (c) a portion of the time behaviour of x from which (b) was calculated. A log-log plot of the spectrum shown in figure 4b shows a slope of -1 in the low frequency range. Time-series characterized by such $1/f$ spectra display long-term memory and low-frequency fluctuations which do not level off with an increase of

Figure 2. Short time-series for the maps shown in figure 1. (a) Time-series of x , (b) time-series of $y = 1 - e^{-4x}$ and (c) time-series of $z = e^{4x}$.

Figure 3. The power spectra of the logistic map and its relatives obtained by changes of variables as in figure 1. The spectra are averages over 100 realizations with different initial conditions.

the length of the record. (See, for example, Halley (1996) for a discussion on the role of $1/f$ processes in ecology and evolution).

Reddened spectra are also obtained when g is deterministically prescribed. Figure 5, with panels as in figure 4,

shows the results for a case where the driver (2) is also given by a logistic equation, namely

$$R(n+1) = 3.9R(n)[1 - R(n)], \quad (5)$$

with $a = 3.98$ and $b = -0.37$ in equation (4). The reddening in this case is distinct, though less so than in the stochastic case; this feature varies with the parameters in equation (5).

Note that, unlike the reddened spectra obtained by choosing special parameters in the model (e.g. White *et al.* 1996*b*; Blarer & Doebeli 1996), the spectra of this study show an increasing power at very low frequencies. Conversely, spectra in White *et al.* (1996*b*) and Blarer & Doebeli (1996) display a dominant broad peak at low frequencies, below which the spectral power decreases.

4. CONCLUSIONS

We have mentioned several mechanisms that may produce red spectra in the output of simple models for population dynamics. The most significant for the real situation would seem to be fluctuations in the environment, which, as indicated by Sugihara, may cause reddening of the power spectrum of population fluctuations. This effect can be achieved even when the spectrum of $R(n)$, the fluctuating parameter representing

Figure 4. Results from equations (2) and (3) with g as a white noise. (a) The power spectrum of $R(n)$; (b) the power spectrum of $x(n)$ and (c) a short time-series for $x(n)$. These power spectra are averages over 100 realizations with different initial conditions.

Figure 5. Power spectra and time-series as in figure 4 but from equations (4) and (5).

environmental variability, is white or blue. Results of this kind have been discussed by Kaitala *et al.* (1997) for stochastic parameter fluctuations. Here, we have shown that both stochastic and deterministic parameter fluctuations can induce spectral reddening when the system is intermittent. In particular, the on-off mechanism, that we have briefly described is robust under slight changes of the control parameter or the addition of weak noise. This mechanism works even when there is a feedback of the population on the environmental variations, with R depending on x ; on-off intermittency can still occur (Spiegel 1981) with consequent spectral reddening. We thus have a simple mechanism that reliably produces bursty time-series with red spectra such as are widely observed in many intermittent processes and

is likely to play a role in many applications of chaotic maps such as models of population dynamics.

We thank Dr John Steele for introducing us to this problem and for his help along the way. We also thank him and Dr Istvan Scheuring for directing us to relevant references. We thank Dr Albert Blarer and an anonymous referee for interesting comments on this work and Dr Blarer for clarifying the difference between his approach and ours. C.T. was partially supported by NSF.

REFERENCES

- Allen, M. R. & Smith, L. A. 1994 Investigating the origins and significance of low-frequency modes of climate variability. *Geophys. Res. Lett.* **21**, 883–886.
- Blarer, A. & Doebeli, M. 1996 In the red zone. *Nature* **380**, 589–590.
- Brockwell, P. J. & Davis, R. A. 1987 *Time series: theory and methods*. Berlin: Springer.
- Cohen, J. E. 1995 Unexpected dominance of high frequencies in chaotic nonlinear population models. *Nature* **378**, 610–612.
- Diamond, J. M. & May, R. M. 1977 Species turnover rates on islands: dependence on census interval. *Science* **197**, 266–270.
- Ding, M. & Yang, W. 1997 Stability of synchronous chaos and on-off intermittency in coupled map lattices. *Phys. Rev. E* **56**, 4009–4016.
- Duane, G. S. 1997 Synchronized chaos in extended systems and meteorological teleconnections. *Phys. Rev. E* **56**, 6475–6493.
- Halley, J. M. 1996 Ecology, evolution and $1/f$ noise. *Trends Ecol. Evol.* **11**, 33–37.
- Kaitala, V. & Ranta, E. 1996 Red/blue chaotic power spectra. *Nature* **381**, 198–199.
- Kaitala, V., Ylikarjula, J., Ranta, E. & Lundberg, P. 1997 Population dynamics and the colour of environmental noise. *Proc. R. Soc. Lond. B* **264**, 943–948.
- Landa, P. S., Zaikin, A. A., Rosenblum, M. G. & Kurths, J. 1998 On-off intermittency phenomena in a pendulum with a randomly vibrating suspension axis. *Chaos, Solitons and Fractals* **9**, 157–169.
- Ma, S. C. 1958 The population dynamics of the oriental migratory locust (*Locusta migratoria manilensis* Meyen) in China. *Acta Ent. Sin.* **8**, 1–40.
- May, R. M. & Oster, G. 1976 Bifurcations and dynamic complexity in simple ecological models. *Am. Nat.* **110**, 573–590.
- Miramontes, O. & Rohani, P. 1998 Intrinsically generated coloured noise in laboratory insect populations. *Proc. R. Soc. Lond. B* **265**, 785–792.
- Pimm, S. L. & Redfearn, A. 1988 The variability of population densities. *Nature* **334**, 613–628.
- Platt, N., Spiegel, E. A. & Tresser, C. 1993a On-off intermittency: a mechanism for bursting. *Phys. Rev. Lett.* **70**, 279–282.
- Platt, N., Spiegel, E. A. & Tresser, C. 1993b The intermittent solar cycle. *Geophys. Astrophys. Fluid Dyn.* **73**, 147–161.
- Spiegel, E. A. 1981 A class of ordinary differential equations with strange attractors. *Ann. NY Acad. Sci.* **357**, 305–312.
- Steele, J. H. 1985 A comparison of terrestrial and marine ecosystems. *Nature* **313**, 355–358.
- Steele, J. H. & Henderson, E. W. 1992 A simple model for plankton patchiness. *J. Plankt. Res.* **14**, 1397–1403.
- Sugihara, G. 1995 From out of the blue. *Nature* **378**, 559.
- von Hardenberg, J., Paparella, F., Platt, N., Provenzale, A., Spiegel, E. A. & Tresser, C. 1997 Missing motor of on-off intermittency. *Phys. Rev. E* **55**, 58–64.
- White, A., Begon, M. & Bowers, R. G. 1996a Red/blue chaotic power spectra. *Nature* **381**, 198.
- White, A., Begon, M. & Bowers, R. G. 1996b Explaining the colour of power spectra in chaotic ecological models. *Proc. R. Soc. Lond. B* **263**, 1731–1737.