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An Efficient Reconstruction Method for Non-Uniform Attenuation Compensation with Non-Parallel Beam Geometries Based on Novikov's Explicit Inversion Formula

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Abstract

This paper investigated an accurate reconstruction method to invert the attenuated Radon transform in non-parallel beam (NPB) geometries. The reconstruction method contains three major steps: (1) performing one-dimensional phase-shift rebinning, (2) implementing nonuniform Hilbert transform, and (3) applying Novikov's explicit inversion formula. The method seemed to be adaptive to different settings of fan-beam geometry from very long to very short focal lengths without sacrificing reconstruction accuracy. Compared to the conventional bilinear rebinning technique, the presented method showed a better spatial resolution, as measured by modulation transfer function. Numerical experiments demonstrated its computational efficiency and stability to different levels of Poisson noise. Even with complicated geometries such as varying focal-length and asymmetrical fan-beam collimations, the presented method achieved nearly the same reconstruction quality of parallel-beam geometry. This work can facilitate quantitative reconstruction of single photon emission computed tomography for cardiac imaging, which may need NPB collimation geometries and require high computational efficiency.

Keywords

Attenuated Radon transform; NPB collimation geometry; phase shifting; nonuniform Hilbert transform

I. Introduction

Quantitative reconstruction of single photon emission computed tomography (SPECT) with compensation for non-uniform attenuation is mathematically formulated as an inversion of the attenuated Radon transform (AtRT) [1]. Recently an explicit inversion formula for the AtRT was derived by Novikov for parallel-beam (PB) collimation geometry using wave propagation and generalized distribution theories [2], and later a simpler derivation of an equivalent formula was given by Natterer [3]. In many clinical applications, however, non-parallel beam (NPB) collimation geometries may be desired, such as fan-beam (FB), asymmetrical fan-beam (AsF), or spatially varying focal-length fan-beam (VFF) geometries [4][5][6]. For FB geometry, an explicit inversion formula was discovered by Bukhgeim and Kazantsev via a very different approach from that of Novikov's as solving the boundary condition problem of a transport equation [7][8]. Their algorithm performs well when a specific equi-angular FB sampling is

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used [7]. However, applying their formula to other geometries, for example an equi-spatial FB sampling geometry, is not straightforward. The challenge becomes even severer for deriving an explicit inversion formula for other more complicated collimation geometries mentioned above.

To the best of our knowledge, there is not yet a unified exact inversion formula for the AtRT in NPB geometries, except for the ray-driven technique we developed previously [9]. This technique is built on the fact that each line integral in the NPB measurements corresponds to a one line integral in the PB dataset. Due to the linearity of the inversion of the AtRT (which can be considered as a weighted summation or filtered backprojection operation), each line integral can be processed separately; and therefore the Novikov's inversion formula for the AtRT in PB geometry can be applied to perform the desired reconstruction in a NPB geometry ray-by-ray. However, the computational cost of this ray-driven technique is considerably high as known as $O(N^4)$, as compared to $O(N^3)$ of the Novikov's inversion formula for PB geometry [3][10].

In this work, we investigated a more efficient reconstruction method for the inversion of the AtRT in NPB geometries, which preserves the same accuracy as its ancestor for PB inversion procedure. This alternative method is based on the Novikov's explicit inversion formula, phase-shift rebinning and non-uniform Hilbert transform (HT). Its derivation is presented for the VFF geometry, of which the FB geometry is a special case. Numerical studies are performed for equi-spatial FB, VFF, and AsF geometries in the presence of strongly non-uniform attenuating media, similar to that occur in the human thorax. Its stability is tested for different focal lengths of FB geometry and different levels of Poisson noise. Its impact on spatial resolution is evaluated by comparing the modulation transfer function (MTF) of different reconstruction methods, which includes the PB inversion formula (implemented in [3], which is found to have a little better spatial resolution than the implementation of [10]), the presented method for NPB geometries, and a conventional rebinning technique using bilinear interpolation (from NPB to PB geometries, so that the PB inversion formula [3] can be used).

The rest of this paper is organized as follows. In Section II, basic notations are introduced for the considered NPB geometries. In Section III, brief reviews of the phase-shift rebinning and the Novikov's inversion formula are presented first, and then extension of the Novikov's inversion formula to NPB geometries is developed by the use of the phase-shift rebinning and the non-uniform HT. Detailed implementation of the presented reconstruction method is also summarized in this section. Finally, numerical simulation results are reported in Section IV to show several properties of the presented reconstruction method.

II. Basic Notations and Non-Parallel Geometries

The VFF geometry for SPECT is shown, as an example, in Figure 1. The acquired datum at point B on the detector coordinate s , is the line integral of radionuclide distribution $f(x, y)$ along line AB , affected by attenuation given by the attenuation coefficient map $\mu(x, y)$, where A is located on the detector focal plane in Figure 1, corresponding to the focal length $F(s)$. The focal point A has a varying distance to the origin. When the focal length increases from a minimum for rays passing through the center to a maximum for rays passing through the edge of the object, function $F(s)$ should increase with the absolute value $|s|$. A functional form can be $F(s) = as^2 + b$ with a and b as constants.

Assume that the initial position of the flat detector is parallel to the x -axis with distance R from the origin. After the detector is rotated around the origin from the initial position by angle β , every projection ray at this position can be uniquely determined by the coordinates (β, s) . Comparing with the PB coordinates (θ, l) , see Figure 1, we have the following relations:

$$\theta = \beta + \tan^{-1}\left[\frac{s}{F(s)}\right] \text{ and } l = \frac{s[F(s) - R]}{\sqrt{s^2 + F^2(s)}}. \quad (1)$$

The Jacobian from coordinates (β, s) (or $d\beta ds$) to coordinates (θ, l) (or $d\theta dl$) is:

$$J(s) = \left[sF'(s)(s^2 + RF(s)) + (F(s) - R)F^2(s) \right] / \left(s^2 + F^2(s) \right)^{3/2}, \quad (2)$$

where $F'(s)$ is the derivative of $F(s)$ with respect to s .

In discrete case, we assume that β is evenly sampled in $[0, 2\pi)$ at M points and s is evenly sampled at N points within the range $[-s_0, s_0]$. In other words, the VFF projections $g(\beta, s)$ are acquired at (β_j, s_k) , where $\beta_j = j\delta_1$, ($j = 0, 1, \dots, M-1$) and $s_k = k\delta_2 - s_0$, ($k = 0, 1, \dots, N-1$) with $\delta_1 = 2\pi/M$ and $\delta_2 = 2s_0/(N-1)$. The sampled points (β_j, s_k) can be mapped to (θ_j, l_k) according to equation (1) and displayed in the sinogram space as Figure 2(a). For comparison, Figure 2(b) shows the sampled points of PB projections in the same space. Obviously, the sampling grid of Figure 2(a) from VFF geometry is less uniform than that of PB geometry of Figure 2(b). It is noticed that for every fixed l_k or s_k , $\{\theta_j\}$ are evenly distributed along the vertical line of the corresponding radial distance for both geometries. Since the sampling rate (M points over 2π angle) or increment (δ_1) along the angular axis are the same for both the PB and NPB geometries, we assume that there is no essential difference between these two sampling schemes at any single radial distance in terms of the information collected, except for a phase shift in the frequency space. This phase-shifting property is explicitly exploited in our reconstruction method derivation in the next section.

A conventional reconstruction method for NPB geometries may employ a bilinear interpolation to resample the non-uniform grid of Figure 2(a) to the uniform grid of Figure 2(b), and then apply the PB inversion formula to reconstruct the re-sampled data on the uniform grid. The drawback of the re-sampling using nearby grids is well known of losing the spatial resolution in reconstructed images. More discussions are given later.

III. Theory

In this section, the basic theory for our reconstruction method and its implementation are presented in VFF geometry without losing the generality.

A. Phase-Shifting Property between PB and NPB Collimation Geometries

Let $g(\beta, s)$ denote the projection data in the VFF geometry, and $p(\theta, l)$ be the projection data in the PB geometry. If a full scan is performed along a circular orbit for each geometry, then $g(\cdot)$ and $p(\cdot)$ are 2π -periodic functions of the view angle β and θ , respectively, and are related, as shown in Figure 1, by the following equation:

$$g(\beta, s) = p\left(\beta + \tan^{-1}\left[\frac{s}{F(s)}\right], \frac{s[F(s) - R]}{\sqrt{s^2 + F^2(s)}}\right). \quad (3)$$

By Fourier transform (FT) on the angular variables, it is easy to show that a phase-shifting relation exists between these two geometries:

$$P_N\left(\frac{s[F(s) - R]}{\sqrt{s^2 + F^2(s)}}\right) = \exp\left[-in \tan^{-1}\left(\frac{s}{F(s)}\right)\right] G_N(s) \equiv \mathcal{G}_N(s), \quad (4)$$

where $i = \sqrt{-1}$, $\mathcal{G}_N(s)$, is defined as *shifted Fourier series*, and

$$P_n(l) = \int_0^{2\pi} p(\theta, l) e^{-in\theta} d\theta \quad \text{and} \quad G_n(s) = \int_0^{2\pi} g(\beta, s) e^{-in\beta} d\beta. \quad (5)$$

With the Fourier coefficients, we can obtain the re-phased projection data:

$$p(\phi, \frac{s[F(s) - R]}{\sqrt{s^2 + F^2(s)}}) = \sum_n G_n(s) \exp(in\phi), \quad (6)$$

in which the angular positions at each radial distance are aligned. These sampled points of the re-phased projection data $p(\phi, \frac{s[F(s) - R]}{\sqrt{s^2 + F^2(s)}})$, derived from the measurements $g(\beta_j, s_k)$, are

illustrated in Figure 3. Now the angular sampling becomes uniform, i.e., the same as that of the PB projection data, although the radial sampling remains non-uniform, which is determined by the sampling function $\frac{s_k[F(s_k) - R]}{\sqrt{s_k^2 + F^2(s_k)}}$, as compared to Figure 2(b) for the uniform radial

sampling in the PB geometry. (Handling the radial non-uniform sampling will be addressed later by the implementation of the non-uniform HT). In practice, the calculations of $G_n(s)$ and $p(\phi, \frac{s[F(s) - R]}{\sqrt{s^2 + F^2(s)}})$ are accomplished by the use of fast FT (FFT) and thus are generally very efficient in computation for the phase shifting.

The above phase-shifting relation is essentially a kind of functional interpolation (or rebinning) using Fourier technique. Mathematically it has superior properties over the local interpolations that use a finite number of nearest-neighbor points for a linear, spline or the like interpolation. In general, the phase-shifting Fourier interpolation has higher order accuracy than the local interpolation, if the underlying source distribution function is smooth [11]. From another point of view, if the source function can be restored from the originally sampling data, which indicates that the sampling rate is sufficiently high, then the re-phased data with the same sampling rate should be able to restore the same function, while the local interpolation may not be able to do so. The optimality of the phase-shifting functional interpolation has actually been utilized implicitly by others for different purposes. For example, P. J. Rivière *et al.* used it in developing reconstruction algorithm for helical CT (computed tomography) [12]; X. Pan *et al.* in controlling the noise property of FB CT image reconstruction [13][14]; and J. You *et al.* in developing the circular harmonic decomposition algorithms for inversion of the Radon transform and the exponential Radon transform under VFF geometry [15][16].

B. Implementation of Novikov's Inversion Formula and Non-uniform HT for NPB Collimation Geometries

In order to present our reconstruction method for NPB geometries, we first briefly describe the Novikov's inversion formula for PB geometry. In Figure 4, a SPECT imaging system with PB collimation geometry is illustrated. The γ photons, emitted from the radiotracer distribution $f(x, y)$, are attenuated inside the body with coefficient distribution $\mu(x, y)$ before they arrive at the detector. Let $p(\theta, l)$ be the accumulated photon counts at position l along projection or detector view angle θ in the PB geometry, then

$$p(\theta, l) = \int_{-\infty}^{\infty} f(l, t) \exp[-a_\theta(l, t)] dt, \quad (7)$$

where $a_{\theta}(l, t) = \int_t^{\infty} \mu_{\theta}(l, \tau) d\tau$, and subscript θ denotes the expression of a function in the rotating coordinate system (l, t) .

Equation (7) is the definition of the AtRT for PB geometry, and the Novikov's inversion formula can be expressed by our notations as follows:

$$f(\vec{r}) = \frac{1}{4\pi} \text{div} \int_0^{2\pi} \int_{l=\vec{r} \cdot \vec{j}}^{\infty} \exp([D\mu]_{\theta}(l, t)) p(\theta, l) \Big|_{t=\vec{r} \cdot \vec{k}} d\theta, \quad (8)$$

where **div** is the divergence operation, $\vec{j} = (\cos\theta, \sin\theta)$, $\vec{k} = (-\sin\theta, \cos\theta)$, and $p(\theta, l)$ is called modified projection data:

$$p(\theta, l) = e^{-h_1} \left\{ \cos(h_2) [H \cos(h_2) e^{h_1} p](\theta, l) + \sin(h_2) [H \sin(h_2) \exp(h_1) p](\theta, l) \right\}, \quad (9)$$

with $h_1 = \frac{1}{2}[\mathbf{R}\mu](\theta, l)$ and $h_2 = [\mathbf{H}h_1](\theta, l)$. Here the operators **H**, **D**, and **R** are defined as follows:

$$\text{Hilbert transform: } [H_g](s) = \frac{1}{\pi} \text{p.v.} \int_{-\infty}^{\infty} \frac{g(\tau)}{s - \tau} d\tau \quad (10)$$

$$\text{Half line integral: } [D\mu]_{\theta}(s, t) = \int_t^{\infty} \mu_{\theta}(s, \tau) d\tau \quad (11)$$

$$\text{Radon transform: } [Rf](s, \theta) = \int_{-\infty}^{\infty} f_{\theta}(s, \tau) d\tau. \quad (12)$$

where p.v. is the Cauchy principle value integral. Detailed derivation of the Novikov's inversion formula can be found in [2][17]. Direct discretization of the inversion formula (8) or so-called generalized filtered backprojection (FBP) algorithm for the AtRT has been reported by selecting different kernels to approximate the operator **H** in equation (10) [3][10]. A classical FBP implementation for a simplified version of formula (8) has been investigated recently by the use of conventional filters, such as Ramp, Hann, Shepp-Logan, etc. [18], where the two steps of filtering the emission data and back-projecting the filtered emission data mimic the conventional FBP procedure [19][20].

The explicit inversion formula (8) is exact for continually sampled data in PB geometry. By a one-dimensional (1D) local interpolation along the radial distance axis, it would be straightforward to apply the inversion formula to the re-phased non-uniform data

$p(\phi, \frac{s[F(s) - R]}{\sqrt{s^2 + F^2(s)}})$ of Figure 3 for image reconstruction. However, the local interpolation

usually would result in the loss of reconstruction resolution [14] and will not be explored in this work. In the following, we will rather retain the non-uniformly sampled projection data

$p(\phi, \frac{s[F(s) - R]}{\sqrt{s^2 + F^2(s)}})$, modify their related weighting functions in the inversion formula (8), and

employ non-uniform HT to ensure an accurate image reconstruction without the conventional 1D interpolation.

Substitute $p(\phi, \frac{s[F(s) - R]}{\sqrt{s^2 + F^2(s)}})$ into formula (8), and after a few calculations one can obtain an inversion formula for the non-uniformly sampled PB projection data p as follows:

$$f(\vec{r}) = \frac{1}{4\pi} \int_0^{2\pi} \frac{\partial}{\partial \xi} \left\{ \exp[\mathcal{D}\mu]_{\phi}(\xi, t) * [\cos(h_1(\phi, \xi)) \mathcal{H}g_1(\phi, \xi) + \sin(h_1(\phi, \xi)) \mathcal{H}g_2(\phi, \xi)] \right\} \Big|_{\substack{\xi = \vec{r} \cdot \vec{j} \\ t = \vec{r} \cdot \vec{k}}} d\phi \quad (13)$$

where

$$\xi = \frac{s[F(s) - R]}{\sqrt{s^2 + F^2(s)}}, \quad h_1(\phi, \xi) = \frac{1}{2} [\mathcal{R}\mu]_{\phi}(\xi), \quad h_2 = [\mathcal{H}h_1](\phi, \xi), \quad g_1(\phi, \xi) = \cos(h_2) \exp(h_1) p(\phi, \xi), \quad (14)$$

$$g_2(\phi, \xi) = \sin(h_2) \exp(h_1) p(\phi, \xi), \quad \text{and} \quad [\mathcal{D}\mu]_{\phi}(\xi, t) = [\mathcal{D}\mu]_{\phi}(\xi, t) - h_1(\phi, \xi).$$

It is noted that equation (13) has been simplified from the original inversion formula (8) by removing the calculations of complex values, in addition to the extension to the non-uniformly sampled PB projection data after the phase-shifting property is considered. In discrete presentation, it is often the case that s is evenly sampled in $[-s_0, s_0]$, as described before, then ξ becomes non-evenly sampled. For functions $g_1(\cdot, \xi)$ and $g_2(\cdot, \xi)$ in equation (13) or $h_1(\cdot, \xi)$ in equation (14), which are sampled on non-uniform grids ξ_i , their discrete HT must be carefully evaluated. One method would be the use of the trapezoidal rule for a functional $g(\xi)$ as follows:

$$[\mathcal{H}g](\xi_i) = \sum_{j \neq i}^N g(\xi_j) \frac{1}{\pi(\xi_j - \xi_i)} \Delta_j, \quad \xi_i = \xi(s_i), \quad \Delta_i = \frac{1}{2}(\xi_{i+1} - \xi_{i-1}). \quad (15)$$

By our previous study [18], however, it is found that the direct implementation of HT via equation (15), which is essentially a convolution with $1/\pi x$ (0 , if $x = 0$), can result in the loss of image resolution. To avoid this possible drawback, we adopt the continuous form of a low-pass filter proposed by Natterer for the HT [3]:

$$[\mathcal{H}_{\Omega}g](\xi) = \int_{-\infty}^{\infty} \frac{1 - \cos(\Omega(\xi - t))}{\pi(\xi - t)} g(t) dt, \quad (16)$$

where $\Omega > 0$ is the bandwidth. Then the HT of a functional $g(\cdot)$ sampled on the non-uniform grid $\xi(s)$ can be derived as:

$$[\mathcal{H}_{\Omega}g](\xi_0) = \int_{-\infty}^{\infty} \frac{1 - \cos(\Omega(\xi_0 - \xi(s)))}{\pi(\xi_0 - \xi(s))} g(\xi(s)) \left[s F'(s) (s^2 + R F(s)) + (F(s) - R) F^2(s) \right] \Big/ (s^2 + F^2(s))^{3/2} ds \quad (17)$$

and a discrete non-uniform version is expressed as follows:

$$[\mathcal{H}_{\Omega}g](\xi_i) = \sum_j \frac{1 - \cos(\Omega(\xi_i - \xi_j))}{\pi(\xi_i - \xi_j)} g(\xi_j) \frac{s_j F'(s_j) (s_j^2 + R F(s_j)) + (F(s_j) - R) F^2(s_j)}{(s_j^2 + F^2(s_j))^{3/2}}. \quad (18)$$

It should be noted that different implementations of the HT will affect the reconstruction property in general. The HT plays a role as a regularizing filter in the classical FBP

reconstruction [18][19][20]. For example, Natterer [3] and Kunyansky [10] used different implementation formulas for HT and obtained different results. Therefore, the extended nonuniform HT formula of (18), derived from equation (16), may be replaced by another form of regularization techniques. In addition, a so-called min-max criterion used in developing the nonuniform FFT (NUFFT) [21][22] could be useful in examining the properties of different forms of non-uniform HT. Detailed investigation on the impact of each form of HT on the reconstruction is out of the scope of this paper.

C. Summary of the Presented Reconstruction Method for NPB Geometries

We summarize the implementation of the presented reconstruction method as follows:

1) Modify the NPB projection data $g(\beta, s)$ to obtain the re-phased data $p(\phi, \zeta)$ by the use of the phase-shifting relation: $p(\phi, \xi) = \sum_n \exp[-in \tan^{-1}(\frac{s}{F(s)})] G_n(s) \exp(in\phi)$, and

$$\xi = \frac{s[F(s) - R]}{\sqrt{s^2 + F^2(s)}};$$

2) Calculate the following terms for the weighted projections, where the non-uniform HT of equation (18) is applied:

$$(i) h_1(\phi, \xi) = \frac{1}{2} [R\mu]_{\phi}(\xi), \quad h_2 = [Hh_1](\phi, \xi), \text{ and } [D\mu]_{\phi}(\xi, t) = [D\mu]_{\phi}(\xi, t) - h_1(\phi, \xi),$$

$$(ii) g_1(\phi, \zeta) = \cos(h_2) \exp(h_1) p(\phi, \zeta) \text{ and } g_2(\phi, \zeta) = \sin(h_2) \exp(h_1) p(\phi, \zeta),$$

$$(iii) m_{\phi}(\xi(s), t) = \exp([D\mu]_{\phi}(\xi, t)) \times [\cos(h_1(\phi, \xi)) [Hg_1](\phi, \xi) + \sin(h_1(\phi, \xi)) [Hg_2](\phi, \xi)];$$

3) Perform the differentiation and backprojection operations for the reconstructed image of

$$f(\vec{r}) = \frac{1}{4\pi} \int_0^{2\pi} \left\{ J(s) \frac{\partial}{\partial s} m_{\phi}(\xi(s), t) \right\} \Big|_{s=\vec{r} \cdot \vec{j}} \Big|_{t=\vec{r} \cdot \vec{k}}, \text{ where the Jacobian is given by equation (2) or } J(s) \\ = [sF'(s)(s^2 + RF(s)) + (F(s) - R)F^2(s)] / (s^2 + F^2(s))^{3/2}.$$

The above derivation is based on the VFF geometry, but it is straightforward to apply the reconstruction formula to other NPB geometries by selecting different focal-length functions $F(s)$. It is clearly seen that the complexity of the implementation for the NPB geometry reconstruction is at the same order as that for the PB case. Numerical studies on the computational efficiency and accuracy, as well as several other properties of the NPB geometry reconstruction, are presented in the next section.

IV. Computer Simulations

In this section, we report computer simulation studies on the presented NPB geometry reconstruction method. First, we show that the method performs well in noiseless simulations for typical FB, VFF, and AsF geometries, where the reconstruction accuracy is demonstrated by comparison of the reconstructed image profiles with that of the digital phantom (ground truth) and by relative error measurements on the reconstructed images. The reconstruction time is also reported. Secondly, without losing the generality, we study several properties of the method under FB geometry. These properties include the stability to the variation of the focal length and the impact on the reconstructed image resolution. Finally, the robustness to the data noise is investigated with VFF geometry.

The emission map used in the simulations is the 2D Shepp-Logan mathematic phantom. The attenuation map contains several ellipse structures with three different attenuation coefficient

values similar to those of water, soft tissues, and bones in human being, see Figure 5. The attenuation coefficients are for the energy 140 keV (Tc-99m). Emission map $f(x, y)$ and attenuation map $\mu(x, y)$ were sampled in a grid of 128×128 . Projections $[\mathbf{R}_{\mu, f}](s, \theta)$ were sampled with 128 bins by 128 views evenly spaced on 360 degrees for a corresponding geometry. Each projection datum was computed by the line integral through the phantom maps. Without specifically stating, the same configurations will be applied in the following simulations.

A. Image Reconstruction for FB, VFF, and AsF Geometries

In the first simulation experiment, we tested the presented reconstruction method for FB geometry. The focal length is 62.5 cm, the radius of rotation (ROR) is 17.5 cm, the detector size/length is 48 cm (evenly sampled by 128 detector bins of size 0.19 cm), and the image pixel size is 0.125 cm. The simulated sinogram and its reconstructed image by the presented method are shown in Figure 6(a). In Figure 6(b), we compare the horizontal profiles of both the reconstructed image and the phantom. Excellent agreement is observed.

In the second experiment, we tested the method for the VFF geometry, where the focal length function is $F(s) = as^2 + b$ with $a = 0.24$, and $b = 45$. The detector size/length remains 48 cm (evenly sampled by 128 detector bins of size 0.38 cm), and the ROR also remains 17.5 cm. The focal length varies from minimum of 45 cm to maximum of 178.24 cm (when $s = 24$ cm). The results are shown in Figure 7(a). The horizontal profiles (drawn at the locations as that in Figure 6) of the reconstructed image and the phantom are compared in Figure 7(b). Excellent agreement is seen.

In the third experiment, the method was tested for the AsF geometry. The AsF configuration is presented in Figure 8. The flat detector of 60 cm is equidistantly sampled by 128 bins (with a size of 0.23 cm). Different from their purpose of increasing the FOV (field-of-view) with data truncation [6], we expanded the detector size to avoid the truncation (the size is slightly larger than the currently available size of 54 cm of Siemens E-CAM system for the purpose of testing the algorithm). The simulated sinogram and its reconstructed image, as well as their horizontal profiles, are shown in Figures 9(a) and 9(b). Excellent agreement is demonstrated.

If the detector size in Figure 8 is reduced, truncation may occur, for example, in the case as illustrated in [6] for the purpose of increasing the FOV of transmission scans in a combined CT/SPECT system. By a fully circular 360° scan, the acquisition from the truncated geometry could completely sample the data even with non-uniform attenuation. This observation has been speculated recently by others [23][24][25], which is one topic of the short-scan problems. Another similar topic is the limited angle scanning geometry rather than the truncation by a small detector. Analytical solutions to these short-scan problems for the inversion of the AtRT are yet to be worked out.

From the above three experiments, the presented reconstruction method demonstrated excellent performance for FB, VFF, and AsF geometries. The horizontal profiles of the reconstructed images match closely to that of the ground truth. For a further comparison, we also plotted the horizontal profiles of the reconstructed image from the Novikov' inversion formula (by Natterer) for the PB geometry [3] and compared them to that of the phantom, see Figure 10. The performance of our NPB geometry reconstruction method is very close to that of the PB inversion formula.

For a quantitative evaluation, we calculated the "total relative error" for each reconstructed image, which is defined as:

$$Error \% = 100 \times \left(\frac{\sum_i |f_i - f_i^0|}{\sum_i f_i^0} \right) \quad (19)$$

where f_i^0 is the intensity of pixel i in the phantom image, and f_i is the intensity of pixel i in the reconstructed image. The calculations were performed in a disk region of interest (ROI) that has the same center of the Shepp-Logan phantom with radius of 64 pixels. All negative values in the ROI were zeroed before the calculation. The results are listed in Table 1, along with the computation time for a 128×128 image array size. (For reconstruction of an image with size 128×128 from 128×128 projection rays, the average computation time is approximately 6 seconds by the presented method).

B. Resolution Comparison with Currently Available Methods

From the above experiments, we observed that the presented method can reconstruct quantitatively accurate and artifacts-free images for NPB geometries. In this section, we show the resolution properties with comparison to the PB inversion formula and the conventional rebinning technique by bilinear interpolation.

First, for verification of each method, SPECT sinograms were simulated numerically for both PB and FB geometries. The detector size/length is 54 cm (for Siemens E-CAM system), the ROR remains 17.5 cm, and the focal length of the FB collimator is 45 cm. Image reconstructions were performed by the Natterer's formula for the PB simulation [3], and by the presented method and rebinning technique for the FB simulation. To see the visual difference, the reconstructed images of the Shepp-Logan phantom are presented in Figure 11. All methods reconstructed excellent images by visual judgment, as expected. However, there is some difference on the object boundaries in the images. This difference is detailed by the study below.

Next, a point source was used to measure the PSF (point spread function) of each reconstruction method and calculate their MTFs by the method described in [26][27]. In this simulation study, a point source of 0.25 mm in diameter was located at an arbitrary position (a pixel position (24, 18) in this study), and the non-uniform attenuation map was the same as shown in Figure 5. The reconstructed PSF 2D images were Fourier transformed, and an averaged MTF (denoted by AMTF) was obtained for each method by averaging the MTFs over 2π angles in the 2D Fourier space. The results are shown in Figure 12.

The resolution improvement from PB to FB geometry is clearly seen with the presented reconstruction method (due to the FB amplification geometry), while the benefit is lost if a bilinear interpolation rebinning technique was applied to transform the FB projection data to PB data and then reconstruct the transformed data by the Natterer's inversion formula.

C. Stability to the Variation of the Focal Length

In this section, another property of the presented reconstruction method – the stability to the variation of the focal length is studied, which reflects the system amplification factor. For this study, the simulations were performed for FB geometry, with the focal length varying from very large (infinity) to very short one. The detector size/length is 48 cm, and the ROR remains 17.5 cm.

The simulated sinograms and corresponding reconstructed images are shown in Figure 13. As the focal length becomes shorter, the reconstructed images remain the same quality. The reconstruction generates artifact-free images within the central FOV region, even if truncation occurs at the FOV periphery (for example, when the focal length reduces to 60 cm and 50 cm

in Figure 13). These results demonstrated the stability of the proposed algorithm as the focal length reduces until data truncation.

D. Robustness to the Noise in Measurements

To include noise in simulation studies, each datum in the simulated noise-free VFF projections (in section A) above was assumed as the mean or average from its multiple measurements. This mean was inputted to a Poisson random-number generator to provide a noisy realization of the datum. By applying the random generator to all the noise-free data elements, a noisy realization of the sinogram was obtained, mimicking a SPECT study in the practice. Three different noise level sinograms were generated by scaling the noise-free data and then adding the Poisson noise. The count levels are approximately, for each view of 128 bins, (1) 30, 000 counts, (2) 10, 000 counts, and (3) 2, 000 counts. The noisy sinograms and the reconstructed images are shown in Figure 14.

Apparently, some details in the reconstructed images disappear due to the Poisson noise. Also, the reconstructed image includes randomized spots across the FOV that are common to FBP or other analytical and iterative reconstructions of Poisson-noise projections without regularization or penalty on the noisy data. Nevertheless, the fairly good compensation for nonuniform attenuation is still observed in these noisy cases. In addition, no obvious artifact is seen with the increase of noise. Accurate treatment of the Poisson noise or extraction of signal from the Poisson noisy sinogram for the inversion formula or reconstruction method is an important issue, and detailed discussion is beyond the scope of this work.

V. Discussions and Conclusions

SPECT is a cost-effective functional imaging modality. It aims to reconstruct an image of the radiopharmaceutical distribution within the tissues and organs. The reconstruction is based on the detection of radiation emitted from the radiopharmaceuticals. Because of physical effects such as attenuation, scatter, system response variation, and poor statistics (low counts), SPECT image is severely degraded comparing to other imaging modalities such as magnetic resonance imaging (MRI) or X-ray CT. Quantitative reconstruction for SPECT requires accurate compensation for the attenuation and other degradation factors. The inversion formulas discussed in this paper provided a mathematically sound and unique solution to the attenuation problem [1], which is not only of theoretical interest, but also has practical significance in the sense that it may lead to quantitative SPECT in the future with additional compensation for other degradation factors, such as PSF variation, scatter, and the Poisson noise [28][29][30][31].

Since the success of the FBP algorithm for inverting the Radon transform with PB geometry by Shepp *et al.* [19], extension to FB, cone-beam and other geometries was then immediately pursued in CT field [20]. This history was repeated for inverting the exponential Radon transform in SPECT imaging when the object has a uniform attenuating property. The success for PB geometry by Tretiak *et al.* [32] lay down a base for the extension to FB [33] and VFF geometries [16]. Recently the same situation occurred again, and we have been challenged by the arbitrarily attenuated Radon transform. Since the discovery of an explicit inversion formula for the AtRT in PB geometry by Novikov [2], great efforts have been devoted to look for new ways or extension to NPB geometries [7][8][9]. Mathematical exactness, computational efficiency, and performance stability are of most concerns.

In this work, we presented a unified reconstruction method for extending PB to NPB collimation geometries. The method re-aligns the NPB projection data along the view direction for all detector bins by the phase-shifting relation and implements appropriately the non-uniform HT, so that the Novikov's inversion formula can be extended accurately and efficiently

from parallel to non-parallel geometries. By numerical simulations we showed that the presented NPB reconstruction method has comparable reconstruction quality and computation efficiency as the PB geometry formula and is robust to the variation of focal length (which is related to the system amplification factor) and the quantum noise in measurements.

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References

1. G. T. Gullberg, The Attenuated Radon Transform: Theory and application in medicine and biology, Ph.D. Dissertation, University of California at Berkeley, CA, 1979.
2. R. G. Novikov, "An inversion formula for the attenuated X-ray transformation", preprint, May 2000.
3. Natterer F. "Inversion of the attenuated Radon transform". *Inverse Problems* 2001;17:113–119.
4. Jaszczak RJ, Li J, Wang H, Coleman RE. "SPECT collimation having spatially variant focusing (SVF)". *J Nucl Medicine* 1992;33(5):891.
5. Zeng GL, Gullberg GT, Jaszczak RJ, Li J. "Fan-beam reconstruction algorithm for a spatially varying focal length collimator,". *IEEE Trans Med Imaging* September 1993;12:575–582.
6. Chang W, Loncaric S, Huang G, Sanpitak P. "Asymmetrical fan beam transmission CT on SPECT system". *Phys Med Biology* 1995;40:913–928.
7. A. A. Bukhgeim and S. G. Kazantsev, "Inversion formula for the fan-beam attenuated Radon transform in a "unit disk", preprint, 2002.
8. Arbuzov EV, Bukhgeim AL, Kazantsev SG. "Two-dimensional tomography problems and the theory of A-analytic functions". *Siberian Advances in Mathematics* 1998;8:1–20.
9. Wen J, Li T, Liang Z. "An analytical inversion of non-uniformly attenuated Radon transform with variable focal-length fan-beam collimators". *IEEE Trans Nucl Science* 2003;50:1541–1549.
10. Kunyansky LA. "A new SPECT reconstruction algorithm based on the Novikov explicit inversion formula". *Inverse Problems* 2001;17:293–306.
11. W. H. Press, S. A. Teukolski, W. T. Vetterling, and B. P. Flannery, *Numerical Recipe in C*, 2nd Edition, Chapters 3 & 4, pp.147, 1997.
12. La Rivi PJ, Pan X. "Favorable noise uniformity properties of Fourier-based interpolation and reconstruction approaches in single-slice helical computed tomography". *Medical Physics* 2002;29(6):943–951. [PubMed: 12094989]
13. Pan X. "Optimal noise control in and fast reconstruction of fan-beam computed tomography image". *Medical Physics* 1999;26:689–697. [PubMed: 10360528]
14. Pan X, Yu L. "Image reconstruction with shift-variant filtration and its implication for noise and resolution properties in fan-beam computed tomography". *Medical Physics* 2003;30(4):590–600. [PubMed: 12722811]
15. You J, Liang Z, Bao S. "A harmonic decomposition reconstruction algorithm for spatially varying focal length collimators". *IEEE Trans Med Imaging* December 1998;17:995–1002. [PubMed: 10048856]
16. You J, Liang Z, Zeng GL. "A unified reconstruction framework for both parallel-beam and variable focal-length fan-beam collimators by a Cormack-type inversion of exponential Radon transform". *IEEE Trans Med Imaging* January 1999;18:59–65. [PubMed: 10193697]
17. Novikov RG. "On the range characterization for the two-dimensional attenuated X-ray transformation". *Inverse Problems* 2002;18:677–700.
18. J. You, T. Li, J. Wen, and Z. Liang, "A classical FBP implementation of Novikov's inversion of the attenuated Radon transform for quantitative SPECT imaging", preprint, 2004.
19. Shepp L, Logan B. "The Fourier reconstruction of a head section". *IEEE Trans Nucl Science* 1974;21:21–43.
20. Gabor Herman, *Image Reconstruction from Projections*, Academic Press, Inc. New York, USA, 1980.

21. Fessler JA. "Nonuniform fast Fourier transforms using min-max interpolation". *IEEE Trans Sig Processing* Feb 2003;51(2):560–574.
22. Matej S, Fessler JA, Kazantsev IG. "Iterative tomographic image reconstruction using Fourier-based forward and back-projectors". *IEEE Trans Med Imaging* April 2004;23(4):401–412. [PubMed: 15084066]
23. Pan X, Kao CM, Metz CE. "A family of pi-scheme exponential Radon transforms and the uniqueness of their inverses". *Inverse Problems* 2002;18:825–836.
24. Pan X, Sidky EY, Kao CM, Zou Y, Metz CE. "Short-scan SPECT imaging with nonuniform attenuation and 3D distance distance-dependent spatial resolution". *Phys Med Biology* 2002;47:2811–2833.
25. Guillaume, Bal. "On the attenuated Radon transform with full and partial measurements". *Inverse Problems* 2004;20:399–418.
26. Nickoloff EL, Riley R. "A simplified approach for modulation transfer function determinations in computed tomography". *Phys Med Biology* 1985;32:437–442.
27. Nickoloff EL. "Measurements of the PSF for a CT scanner: appropriate wire diameter and pixel size". *Phys Med Biology* 1988;33:149–155.
28. Li T, Wen J, Liang Z. "Analytical compensation for spatially variant detector response in SPECT with varying focal-length fan-beam collimators". *IEEE Trans Nucl Science* June 2003;50:398–404.
29. Li X, Han G, Lu H, Li L, Liang Z. "A new scatter estimation method using triple window acquisition to fit energy spectrum". *J Nucl Medicine* 2001;42:194.
30. T. Li, X. Li, J. Wang, J. Wen, H. Lu, J. Hsieh, and Z. Liang, "Nonlinear sinogram smoothing for low-dose X-ray CT", *IEEE Trans. Nucl. Science*, in press, 2004.
31. Li T, Wen J, Han G, Lu H, Liang Z. "Evaluation of an efficient compensation method for quantitative fan-beam brain SPECT reconstruction". *IEEE Trans Med Imaging* 2005;24:170–179. [PubMed: 15707243]
32. Tretiak OJ, Metz CE. "The exponential Radon transform". *SIAM J Appl Math* 1980;39:341–354.
33. Weng Y, Zeng GL, Gullberg GT. "Analytical inversion formula for uniformly attenuated fan-beam projections". *IEEE Trans Nucl Science* 1997;44:243–249.

Figure 1.

Two-dimensional presentation of a varying focal-length fan-beam geometry. The geometry remains the same along the z-axis.

Figure 2.

Illustration of the sampling grids in sinogram space for (a) a VFF-beam geometry, and (b) a parallel-beam geometry. Each dot represents the location of one corresponding projection ray in the sinogram space.

Figure 3.

Sampling grids in the sinogram space for the VFF projections of Figure 2 after re-phasing. The sampling along the view angle axis becomes uniform as in parallel-beam geometry, while the sampling along the radial distance axis remains non-uniform.

Figure 4.
Two-dimensional presentation of a parallel-beam detection system geometry.

Figure 5.
Two-dimensional presentation of the emission phantom (left) and the attenuation map (right).

Figure 6(a).
The simulated sinogram (left) and its reconstructed image (right) for fan-beam geometry (with a regular focal length of 62.5 cm).

Figure 6(b).

The horizontal profiles of the fan-beam reconstruction, where the dotted line is the profile of the reconstructed image and the solid line is the profile of the phantom. Left profiles are through the central horizontal line and right ones are across the three tiny objects.

Figure 7(a).

The simulated sinogram (left) and its reconstructed image (right) for the VFF geometry with a quadratic form of focal length $F(s) = as^2 + b$, where $a = 0.24$, and $b = 45$.

Figure 7(b).

The horizontal profiles of the VFF reconstruction, where the dotted line is the profile of the reconstructed image and the solid line is the profile of the phantom. Left profiles are through the central horizontal line and right ones are across the three tiny objects.

Figure 8. Asymmetrical fan-beam geometry. The geometry is the same along the z -axis.

Figure 9(a).

The simulated sinogram (left) and its reconstructed image (right) for the AsF-beam geometry as illustrated in Figure 8.

Figure 9(b).

The horizontal profiles of the AsF reconstruction, where the dotted line is the profile of the reconstructed image and the solid line is the profile of the phantom. Left profiles are through the central line and right ones are across the three tiny objects.

Figure 10.

The horizontal profiles of the parallel-bam reconstruction, where the dotted line is the profile of reconstructed image and the solid line is the profile of the phantom. Left profiles are through the central line and the right ones are across the three tiny objects.

Figure 11.

The reconstructed images from the presented non-parallel beam method (left), the rebinning technique (middle), and the parallel-beam formula (right).

Figure 12.

The average MTF for different reconstruction methods, where the dotted line is from the presented non-parallel reconstruction method, the dash-dotted line is from the rebinning technique for the same non-parallel geometry, and the solid line is from the parallel-beam geometry Natterer's inversion formula.

Figure 13.

The simulated sinograms (top) and their reconstruction results (bottom) for fan-beam geometry with different settings of the focal length.

Figure 14.

The simulated sinograms with different levels of Poisson noise (top) and their corresponding reconstructed images (bottom) under VFF geometry. (a) 1M counts per view of 128 bins, (b) 100K counts per view of 128 bins, and (c) 10K counts per view of 128 bins.

Table 1

Relative error and computation time on a PC Pentium III with 550 Mhz CPU and 128 MB RAM.

<i>Geometry</i>	<i>FB</i>	<i>VFF</i>	<i>AsF</i>	<i>PB</i>
Relative Error (%)	9.28	8.10	9.77	9.94
Computation Time (s)	3.0	3.3	3.2	2.8