ON THE FALSIFIABILITY OF MATCHING THEORY

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Herrnstein's matching theory requires the parameter, k , which appears in the single-alternative form of the matching equation, to remain invariant with respect to changes in reinforcement parameters like magnitude or immediacy. Recent experiments have disconfirmed matching theory by showing that the invariant- k requirement does not hold. However, the theory can be asserted in a purely algebraic form that does not require an invariant k and that is not disconfirmed by the recent findings. In addition, both the original and the purely algebraic versions of matching theory can be asserted in forms that allow for commonly observed deviations from matching (bias, undermatching, and overmatching). The recent finding of a variable k does not disconfirm these versions of matching theory either. As a consequence, matching remains a viable theory of behavior, the strength of which lies in its general conceptualization of all behavior as choice, and in its unified mathematical treatment of single- and multialternative environments.

Key words: matching, choice, quantitative law of effect, bias, undermatching

According to Herrnstein's (1970) matching theory, all behavior is choice and is governed by contingent reinforcement evaluated relative to all reinforcement obtained in a particular environment. This theory unifies the treatment of single- and multialternative environments by asserting that behavior is governed by the same matching equation in both cases (de Villiers, 1977; Herrnstein, 1970). Because of its breadth and importance, the single-alternative form of the matching theory has been referred to as a quantitative statement of the law of effect (e.g., de Villiers, 1977; Herrnstein, 1970; McDowell, 1982).

Herrnstein (1974) showed that matching theory can be falsified by demonstrating that the parameter, k , which appears in the single-alternative form of the matching equation, varies with reinforcement parameters like magnitude or immediacy (cf. de Villiers & Herrnstein, 1976). Such a finding was reported in two recent papers, the authors of which concluded that Herrnstein's account had

been disconfirmed (McDowell & Wood, 1984, 1985; see also de Villiers, 1977; de Villiers & Herrnstein, 1976; McDowell, 1980; McSweeney, 1975). But matching theory can be considered from a perspective that does not require a constant k . Herrnstein's perspective, which will be referred to as the quasi-formal theory of matching, is a set of ideas about behavior and reinforcement, some of which are expressed in mathematical form. A second perspective is produced by disregarding all nonmathematical elements of the quasi-formal theory. The result, which will be referred to as the formal theory of matching, is a purely algebraic account.

Matching theory loses little generality when stripped of its nonmathematical elements. Although restricted to well defined response and reinforcement quantities, the formal theory nevertheless asserts that all behavior is choice governed by the matching equation, and it still unifies the treatment of single- and multialternative environments. The two perspectives differ, however, with respect to their falsifiability. It can be shown that the finding of a variable k falsifies the quasi-formal theory, but not the formal theory of matching. This means that matching theory, when properly asserted, is not seriously damaged by McDowell and Wood's (1984, 1985) finding that the constant- k requirement does not hold.

Herrnstein's matching theory and its constant- k requirement are discussed in the first section of the present paper. In the second

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section, the purely algebraic version of matching theory is discussed and its falsifiability is examined. Both versions of matching theory can also be asserted in weak forms that permit certain deviations (Baum, 1974) from empirical matching. These forms are discussed, and their falsifiability is examined, in the third section of the paper.

THE QUASI-FORMAL THEORY OF MATCHING

The matching equation for a three-alternative environment is written

$$\frac{R_1}{R_1 + R_2 + R_e} = \frac{r_1}{r_1 + r_2 + r_e}, \quad (1)$$

where the R s represent response rates and the r s represent reinforcement rates. The numerical subscripts refer to each of two instrumental response alternatives. The quantity r_e represents the rate of background reinforcement, which is not under experimental control, and R_e represents behavior (e.g., preening) that is supported by the background reinforcement. Neither of these "extraneous" rates can be measured directly (Herrnstein, 1970). The quasi-formal theory is based on Equation 1, written for a two-alternative environment. Letting R_2 and r_2 go to zero, letting $R_1 + R_e = k$, and solving Equation 1 for R_1 produces

$$R_1 = k \left(\frac{r_1}{r_1 + r_e} \right). \quad (2)$$

As a purely algebraic object, this equation states a relationship between a response rate and two reinforcement rates (r_1 and r_e). In the quasi-formal theory, however, Equation 2 is interpreted abstractly as a law of response strength (de Villiers & Herrnstein, 1976; Herrnstein, 1970). According to this interpretation, Equation 2 asserts that response strength is a function of relative reinforcement, which is the factor in parentheses on the right. Response rate is just a measure of response strength in the quasi-formal theory. Latency to respond in a discrete trial situation or running speed in an alley can also be used as measures of response strength. In addition, the theory asserts that response strength is a function of relative reinforcement, not of relative reinforcement rate. The description of

the independent variable is explicitly abstract because factors other than rate can affect the relativity of reinforcement. For example, reinforcer magnitudes can be substituted for the r s in Equation 2 to provide a different kind of relativity—namely, relative magnitude or amount of reinforcement. By changing reinforcement delays, a third kind of relativity can be arranged. Response strength and relative reinforcement are the two principal nonmathematical concepts entailed by the quasi-formal theory. A variety of measures of these two quantities, including the ones mentioned above, have been used to test Equation 2 as a law of response strength (de Villiers & Herrnstein, 1976).

Equation 2 is a somewhat misleading statement of the quasi-formal theory because it suggests that reinforcement rate is the independent variable. The actual independent variable is relative reinforcement. Letting this quantity (which appears in parentheses on the right of Equation 2) equal r_r , Equation 2 can be written

$$R = kr_r, \quad (3)$$

where R , with the subscript dropped, now represents response strength. Equation 3 reveals the Hullian character of the quasi-formal theory of matching. Like Hull's (1943) excitatory potential, drive, or habit strength, Herrnstein's response strength and relative reinforcement are abstract quantities, appropriate measures of which are simply asserted.

Equation 3 expresses response strength as a linear function of relative reinforcement, which is a dimensionless quantity that varies between 0 and 1. Any appropriate measure of response strength, such as rate, latency, or speed, can be used in the equation, and relative reinforcement can be varied by changing any of a number of reinforcement parameters including rate, magnitude, or immediacy.

As Herrnstein (1974) and others (de Villiers and Herrnstein, 1976; McDowell, 1980) have argued, the parameter, k , in Equation 3 is the key to the falsifiability of the quasi-formal theory. The solid line in Figure 1 is a plot of Equation 3. The line has an intercept of zero and a slope equal to k . The figure also shows that when relative reinforcement equals unity, which occurs when all the reinforcement in the environment is obtained from instrumental responding (i.e., when $r_e = 0$ in

Fig. 1. A plot of the mathematical statement of the quasi-formal theory of matching.

Equation 2), the strength of the instrumental response is maximal and equal to k .

Because response strength is an exclusive function of relative reinforcement (according to Equation 3), the method of arranging values of r_r has no bearing on the level of responding that those values will support. A particular r_r , whether arranged by manipulating reinforcement rate, reinforcer magnitude, or some combination of the two, must support the level of responding required by Equation 3. Graphically, this means that the data point must fall on the solid line in Figure 1. Now consider the following case: A certain level of instrumental responding is supported by a certain relative reinforcement rate. If we now double the magnitude of the reinforcer without altering its rate of delivery, a larger proportion of the total number of reinforcers in the environment will be obtained for the instrumental responding. The effect is to change r_r , which will now lie farther to the right on the abscissa of Figure 1. But this new value of r_r must nevertheless support the level of responding required by Equation 3 and the new data point must still fall on the solid line in Figure 1, although it will fall farther to the right. Now it is clear that the slope, k , of this line is the same for all values of r_r . Because the change in reinforcer magnitude described above simply arranges a new value of r_r , k must remain the same when this type of change occurs. Thus, k must remain invariant across changes in reinforcer magnitude and, indeed, across any change in the parameters

of reinforcement that merely alters the value of r_r .

McDowell and Wood's (1984, 1985) experiments arranged changes similar to the one described above. They established a variety of r_r s by manipulating reinforcement rate, and then repeated the experiment several times with different reinforcer magnitudes. They found that k did not remain the same, but instead, increased with increasing reinforcer magnitude. Rather than falling on a single line, the data points fell on a family of lines radiating from the origin, the slopes of which varied directly with the magnitude of the reinforcer. This means that response strength was found not to be an exclusive function of relative reinforcement, and that Equation 3 was shown to be false.

THE FORMAL THEORY OF MATCHING

A purely algebraic, or formal, theory of matching begins with the same matching equation—namely, Equation 1. The first row of Table 1 shows this formal statement of the matching principle written for each of the three response alternatives in a three-alternative environment. The equations in this table will be referred to by their row and column numbers. For example, the equation in the fifth row, second column will be referred to as Equation 5-2.

The 21 equations in Table 1 constitute the entirety of the formal theory of matching. There are no additional assumptions, interpretations, or requirements of any kind. The quantities in the equations are response and reinforcement rates specifically, not measures of response strength or relative reinforcement. Equalities may be preserved in any algebraically legitimate way. Assertions about response allocation or strength, or about the effects of reinforcement, or about any sort of invariance have no bearing on the formal theory. Regarding the last point, notice that the k s in the last row of the table are merely restatements of the response rate sums that appear in the first row. Although these sums are not subject to any special invariance restriction, it is important to recognize that because the sums are identical, the three k s are equal.

The 21 equations in Table 1 are internally consistent. Those in the last 6 rows are de-

Table 1
The 21 matching equations for an ordinary concurrent schedule.

Row	Alternative 1	Alternative 2	Extraneous alternative
1.	$\frac{R_1}{R_1 + R_2 + R_e} = \frac{r_1}{r_1 + r_2 + r_e}$	$\frac{R_2}{R_1 + R_2 + R_e} = \frac{r_2}{r_1 + r_2 + r_e}$	$\frac{R_e}{R_1 + R_2 + R_e} = \frac{r_e}{r_1 + r_2 + r_e}$
2.	$\frac{R_1}{R_1 + R_2} = \frac{r_1}{r_1 + r_2}$	$\frac{R_2}{R_1 + R_2} = \frac{r_2}{r_1 + r_2}$	$\frac{R_e}{R_1 + R_e} = \frac{r_e}{r_1 + r_e}$
3.	$\frac{R_1}{R_1 + R_e} = \frac{r_1}{r_1 + r_e}$	$\frac{R_2}{R_2 + R_e} = \frac{r_2}{r_2 + r_e}$	$\frac{R_e}{R_2 + R_e} = \frac{r_e}{r_2 + r_e}$
4.	$\frac{R_1}{R_2 + R_e} = \frac{r_1}{r_2 + r_e}$	$\frac{R_2}{R_1 + R_e} = \frac{r_2}{r_1 + r_e}$	$\frac{R_e}{R_1 + R_2} = \frac{r_e}{r_1 + r_2}$
5.	$\frac{R_1}{R_2} = \frac{r_1}{r_2}$	$\frac{R_2}{R_1} = \frac{r_2}{r_1}$	$\frac{R_e}{R_1} = \frac{r_e}{r_1}$
6.	$\frac{R_1}{R_e} = \frac{r_1}{r_e}$	$\frac{R_2}{R_e} = \frac{r_2}{r_e}$	$\frac{R_e}{R_2} = \frac{r_e}{r_2}$
7.	$R_1 = \frac{kr_1}{r_1 + r_2 + r_e}$	$R_2 = \frac{kr_2}{r_1 + r_2 + r_e}$	$R_e = \frac{kr_e}{r_1 + r_2 + r_e}$

rived from the mathematical statements of the matching principle that appear in the first row. These derivations are given in the Appendix. Notice that the denominators of the response rate ratios in the first row contain all three response rates. In rows 2 through 4 these denominators contain all possible pairs of response rates. In rows 5 and 6 they contain all possible single response rates except the identities ($R_1/R_1 = r_1/r_1$, etc.), which were omitted. In the last row these denominators contain no response rates. Some equations appear more than once in Table 1. The equations in the first and last rows are identical, inasmuch as k is not assumed to have any special properties. Also, Equations 5-1 and 5-2 are identical, as are Equations 6-1 and 5-3, and Equations 6-2 and 6-3.

As a purely algebraic object, Equation 1-1 preserves most of the generality of the quasi-formal theory. All behavior is choice governed by the matching equation, and the environment with a single instrumental alternative is encompassed by letting R_2 and r_2 go to zero. The difference is that, in the formal theory, these implications apply specifically to rates of responding and rates of reinforcement.

In a purely algebraic sense, Equation 1-1 says nothing about reinforcer magnitude or, indeed, about any parameter of reinforcement other than rate. According to the formal theory, changes in reinforcer magnitude may or may not affect the response rates that appear in Equation 1-1. If they do, the three response rates nevertheless must be related to the three reinforcement rates in the manner specified by the equation.

Consider now the environment with a single instrumental alternative. Letting R_2 and r_2 go to zero in the equations in the first and last rows of Table 1, the second alternative's equations vanish and we are left with

$$\frac{R_1}{R_1 + R_e} = \frac{r_1}{r_1 + r_e} \quad \text{or} \quad R_1 = \frac{kr_1}{r_1 + r_e} \quad (4)$$

for the instrumental alternative, and

$$\frac{R_e}{R_1 + R_e} = \frac{r_e}{r_1 + r_e} \quad \text{or} \quad R_e = \frac{kr_e}{r_1 + r_e} \quad (5)$$

for the extraneous alternative. Suppose now that the magnitude of the instrumental alternative's reinforcer is doubled and that we obtain McDowell and Wood's (1984, 1985) re-

sult—namely, the k in Equation 4 increases. The formal theory permits this outcome provided two conditions are satisfied. First, the equalities in Equations 4 and 5 must be preserved, and second, the k s in the two equations must remain equal. Regarding the first condition, because the change in magnitude increased k , or $R_1 + R_e$, the theory requires that the other quantities in the equations adjust so that the equalities continue to hold. If estimates of k and r_e in the right-hand version of Equation 4 can be found such that the equation describes the data well, then a violation of the equality in Equation 4 was not detected. McDowell and Wood in fact found good fits of Equation 4 regardless of reinforcer magnitude. The existence of satisfactory estimates of k and r_e , however, does not necessarily mean that the real response and reinforcement rates adjusted as required. This can be established only by direct, independent measurement of R_e and r_e which, as noted earlier, is not possible. Good fits of the right-hand version of Equation 4 simply demonstrate stochastically the existence of acceptable R_e and r_e values to which the real extraneous rates could have adjusted, given the observed R_1 and r_1 values. Remember that the objective is not to confirm the theory, but to falsify it. Falsifying the theory on the grounds that the equality condition is violated would entail demonstrating either that the actual rates did not adjust as required, or that acceptable values to which the actual rates could have adjusted do not exist. The former is not possible to demonstrate, and McDowell and Wood's results failed to demonstrate the latter. If a violation of the equality in Equation 4 was not detected, then it follows tautologically that a violation of the equality in Equation 5 was not detected either. The reason for this is that the sum of the two equations is an identity.

The second condition that must be satisfied when reinforcer magnitude changes is that the k s in Equations 4 and 5 must remain equal. If the k in Equation 4 increased, as we have supposed, then the k in Equation 5 must have increased by the same amount because the two k s represent the same sum. A violation of this equal- k condition can be detected, provided independent estimates of the two k s are available. But the k in Equation 5 cannot be estimated independently because R_e and r_e cannot be measured directly. Consequently, in the

environment described by Equations 4 and 5, a violation of the equal- k condition cannot be detected.

To summarize, given an environment with a single instrumental alternative, and given a change in the magnitude of the instrumental alternative's reinforcer, the formal theory may be falsified by showing either that the equalities in Equations 4 and 5 were not preserved or that the k s did not remain equal. McDowell and Wood (1984, 1985) obtained neither of these outcomes and consequently, their results do not falsify the formal theory of matching.

As this example shows, the invariant- k requirement of the quasi-formal theory is replaced by an equal- k requirement in the formal theory of matching. Consider again the first and last rows of Table 1. As noted earlier, the same response-rate sum is represented by the three k s in the last row. If one k increases with reinforcer magnitude, the theory requires only that the other two k s increase by the same amount, and that the various rates adjust so that all of the equalities are preserved. In other words, the formal theory must hold at any given reinforcer magnitude, but it is not required to hold across magnitudes. The reason for this difference is that reinforcer magnitude affects the independent variable in the quasi-formal theory but not in the formal theory of matching.

If one wishes to falsify the formal theory, a situation that arranges only one instrumental response alternative is obviously a poor choice of environment. As demonstrated above, only unacceptable fits of Equation 4 can falsify the theory in this situation. An environment that arranges more than one instrumental response alternative is a better choice because it permits violations of the equal- k requirement to be detected. An ordinary two-alternative concurrent variable-interval variable-interval (VI VI) schedule is the simplest example of this type of environment. A satisfactory procedure would entail arranging a variety of concurrent VI VI schedules by varying the mean intervals of both VI components. This would produce a range of R_1-r_1 pairs and a range of R_2-r_2 pairs, from which independent estimates of the two k s in Equations 7-1 and 7-2 could be obtained. The same procedure would then be repeated a number of times, but with different reinforcer magnitudes arranged for respond-

ing on, say, the first alternative. Based on McDowell and Wood's (1984, 1985) findings, we would expect the k in Equation 7-1 to increase with the magnitude of the reinforcer. But if the independently estimated k in Equation 7-2 did not also increase, and in exactly the same way, then the equal- k condition would be violated, and the formal theory would be disconfirmed. This experiment has not been performed, but McSweeney, Melville, and Whipple (1983) have examined some of the properties of Equations 7-1 and 7-2 when applied to data from concurrent schedules. They concluded that the equations poorly described the data from some studies and that Herrnstein's interpretation of the parameters was implausible in some cases.

To summarize, McDowell and Wood's (1984, 1985) recent finding of a variable k falsifies Herrnstein's version of matching theory, but not the strictly algebraic version of matching theory. The purely formal theory, which retains most of the generality of its quasi-formal predecessor, can be falsified by demonstrating a violation of its equal- k requirement in an ordinary concurrent schedule, although no such demonstration has appeared to date.

WEAK FORMS OF THE QUASI-FORMAL AND FORMAL THEORIES

Neither the quasi-formal nor the formal theory of matching accounts for bias, undermatching, or overmatching. These phenomena have been widely discussed, most recently by Baum (1979) and by Wearden and Burgess (1982). Bias is a constant proportional preference for one of two concurrently available response alternatives that exceeds the preference required by matching. Undermatching describes responding that is nearer indifference than required by matching, and overmatching describes responding that is nearer exclusive preference than required by matching. Although these phenomena actually represent violations of matching theory, they are often regarded as anomalies that must be removed or corrected for in order to reveal the true state of affairs, which is presumed to be matching (cf. Baum, 1979; de Villiers, 1977; Rachlin, 1971).

Following Staddon's (1968) original pro-

posal, concurrent anomalies are typically handled by inserting two parameters in Equation 5-1 as follows:

$$\frac{R_1}{R_2} = b \left(\frac{r_1}{r_2} \right)^a \quad (6)$$

The parameter b represents bias, and the parameter a represents under- or overmatching, which are mutually exclusive. In the absence of any anomalous phenomena, $b = a = 1$, and Equation 6 reduces to Equation 5-1, which is matching. When these phenomena are present, however, they distort the matching relationship and a method like the one specified by Equation 6 must be used to accommodate them.

Equation 6, which has been especially successful in describing concurrent performances (Baum, 1979), can be used to generate weak versions of the quasi-formal and formal theories of matching. These versions are referred to as weak for two reasons: First, they permit violations of strict matching and, second, their presumption that strict matching (Equation 5-1) holds in the absence of distorting phenomena is not subject to disconfirmation (Rachlin, 1971). Nevertheless, Equation 6, which might be referred to as anomalous matching, can be falsified by showing that the equality it requires is violated. If stochastic estimates of a and b that permit a good fit of Equation 6 cannot be obtained, then the equality does not hold and the equation is shown to be false. As already noted, however, estimates of a and b that yield excellent fits of Equation 6 are the rule.

Although their complete development is elementary, the weak forms of the quasi-formal and formal theories of matching will be developed here only to the extent necessary to examine their bearing on the falsifiability of matching theory.

Weak Form of the Quasi-Formal Theory

The equations in Table 1 constitute the mathematical portion of the quasi-formal theory of matching. Considering only the equations in the first column of the table, and letting R_2 and r_2 go to zero in these equations, we are left with four unique expressions—Equations 1-1, 3-1, 6-1, and 7-1, where $R_2 = r_2 = 0$ in the first and last of these. By analogy

with Equation 6, a weak form relevant to the quasi-formal theory may be obtained from Equation 6-1:

$$\frac{R_1}{R_e} = b \left(\frac{r_1}{r_e} \right)^a. \quad (7)$$

The problem is to produce a weak form of Equation 2 from this expression, which can be written

$$\frac{R_1}{R_e} = \frac{br_1^a}{r_e^a}.$$

Inverting, and adding 1 to both sides produces

$$\frac{R_e}{R_1} + \frac{R_1}{R_1} = \frac{r_e^a}{br_1^a} + \frac{br_1^a}{br_1^a},$$

which, after combining terms and reinverting, becomes

$$\frac{R_1}{R_1 + R_e} = \frac{br_1^a}{br_1^a + r_e^a}.$$

This is the weak form of Equation 1-1 where R_2 and r_2 have gone to zero. Letting $R_1 + R_e = k$, and solving for R_1 produces

$$R_1 = k \left(\frac{br_1^a}{br_1^a + r_e^a} \right), \quad (8)$$

which is the desired weak form (cf. Davison & Hunter, 1976).

When there are no concurrent anomalies, $a = b = 1$ and Equation 8 reduces to its strong version, Equation 2. Notice that the k s in the two equations are identical, both being equal to the sum, $R_1 + R_e$.

Just as Equation 2 was interpreted abstractly in the quasi-formal theory, Equation 8 may be interpreted abstractly as a law of response strength. Only now the independent variable, relative reinforcement (which appears in parentheses in Equation 8), is adjusted by the parameters b and a to correct for possible concurrent anomalies.

The invariant- k requirement applies to the weak form of the theory just as it applied to the strong form. Because Equation 8 simply asserts that response strength is a linear function of relative reinforcement, the earlier discussion of the invariant- k requirement applies to Equation 8 as well as to Equation 2. The weak form of the theory, however, adds the

proviso that the independent variable must be corrected for possible concurrent anomalies before the invariant- k requirement can be examined properly. It follows that McDowell and Wood (1984, 1985) did not falsify the weak form of the theory because they ignored anomalous phenomena that could have been responsible for the changes in k that they observed (cf. McLean & White, 1983). To detect a violation of the invariant- k requirement in the weak form of the theory requires data sets from which unique estimates of a , b , k , and r_e in Equation 8 can be obtained. Estimates of k so obtained must remain invariant because a and b were allowed to assume values not equal to 1.

Weak Form of the Formal Theory

Developing the relevant portion of the formal theory in its weak form is slightly more complicated. The goal is to produce weak forms of Equations 7-1 and 7-2. Weak forms analogous to Equation 6 are obtained from Equations 5-2 and 5-3:

$$\begin{aligned} \frac{R_e}{R_1} &= \frac{1}{b_{1e}} \left(\frac{r_e}{r_1} \right)^{a_{1e}}, \\ \frac{R_2}{R_1} &= \frac{1}{b_{12}} \left(\frac{r_2}{r_1} \right)^{a_{12}}, \end{aligned}$$

and

$$\frac{R_1}{R_1} = \frac{r_1}{r_1}.$$

The first equation is the weak form of Equation 5-3. The subscripts on a and b indicate that these parameters pertain to the first and third (or extraneous) response alternatives. The bias parameter is written as a reciprocal so that b_{1e} will be identical to the b -parameter in Equation 7. The second equation is the weak form of Equation 5-2, with appropriate subscripts on the parameters. The third equation is an identity, which was omitted from Table 1. Obviously, $a = b = 1$ for this identity. Summing the three equations produces

$$\frac{R_1 + R_2 + R_e}{R_1} = \frac{1}{b_{1e}} \left(\frac{r_e}{r_1} \right)^{a_{1e}} + \frac{1}{b_{12}} \left(\frac{r_2}{r_1} \right)^{a_{12}} + 1,$$

where r_1/r_1 has been replaced by 1. The reciprocal of this expression is the weak form of

Equation 1-1. Inverting, letting $R_1 + R_2 + R_e = k$, and solving for R_1 yields

$$R_1 = k \left[\frac{1}{b_{1e}} \left(\frac{r_e}{r_1} \right)^{a_{1e}} + \frac{1}{b_{12}} \left(\frac{r_2}{r_1} \right)^{a_{12}} + 1 \right]^{-1}, \quad (9)$$

which is the desired weak form of Equation 7-1. The weak form of Equation 7-2 is produced in exactly the same way, beginning with the identity, $R_2/R_1 = r_2/r_1$, and weak forms of Equations 6-3 and 5-1 with appropriately subscripted parameters. The result is

$$R_2 = k \left[\frac{1}{b_{2e}} \left(\frac{r_e}{r_2} \right)^{a_{2e}} + b_{12} \left(\frac{r_1}{r_2} \right)^{a_{12}} + 1 \right]^{-1}. \quad (10)$$

Notice that the k s in Equations 9 and 10 represent the same response rate sum. Also, when R_2 and r_2 go to zero, Equation 10 vanishes and Equation 9 reduces to Equation 8.

Three distinct a -parameters and three distinct b -parameters appear in Equations 9 and 10. Each of the three sets of parameters pertains to one of the three response-alternative pairs. Furthermore, given r_e and the six a - and b -parameters, it must be the case that

$$\frac{1}{b_{1e}} \left(\frac{r_e}{r_1} \right)^{a_{1e}} b_{12} \left(\frac{r_1}{r_2} \right)^{a_{12}} \frac{1}{b_{2e}} \left(\frac{r_2}{r_e} \right)^{a_{2e}} = 1 \quad (11)$$

for all values of r_1 and r_2 , otherwise the matching principle is violated. Recall that according to matching, the proportion of responses allocated to each alternative in a multialternative environment equals the proportion of reinforcements obtained from that alternative. This principle is expressed mathematically by the equations in the first row of Table 1. Because the three response proportions must sum to 1 (in a three-alternative environment), and because the three reinforcement proportions must also sum to 1, it is obvious that the three equations in the first row of Table 1 must sum to the identity, $1 = 1$; otherwise, the matching principle is violated. In the weak form of the formal theory, the first-row equations sum to this identity only if the condition given in Equation 11 is satisfied. Equation 11 is derived, and its necessity is demonstrated, in the Appendix.

The weak version of the formal theory, like its strong version, requires the k s in Equations 9 and 10 to remain equal, but it does not require an invariant k . This is because the k s

in Equations 9 and 10 represent the same sum, which is not subject to any special invariance restriction when the two equations are considered from a purely algebraic perspective. It follows that McDowell and Wood's (1984, 1985) finding of a variable k does not falsify the formal theory of matching in its weak form. Furthermore, the finding of unequal k s in Equations 7-1 and 7-2, which would falsify the strong version of the formal theory, would not falsify its weak version. According to the weak version, the inequality could be an artifact of concurrent anomalies, which Equations 7-1 and 7-2 ignore (cf. McLean & White, 1983). A proper test of the weak theory requires data sets from which unique estimates of the parameters in Equations 9 and 10 can be obtained. Estimates of k so obtained must remain equal because the a - and b -parameters were allowed to assume values not equal to 1. In this test of the weak theory, it is also necessary to demonstrate that the equality in Equation 11 is not violated. Thus, the weak form of the formal theory may be falsified by obtaining appropriate data sets in the concurrent environment described in the last section. If a violation of any one of the equalities in Equations 9 through 11 is detected, or if a violation of the equal- k condition entailed by Equations 9 and 10 is detected, then the theory would be disconfirmed.

CONCLUSION

Four versions of matching theory have been discussed in this paper and the falsifiability of each has been examined. It was shown that McDowell and Wood's (1984, 1985) finding of a variable k falsifies only the strong form of the quasi-formal theory. This theory survives in its weak form, and the purely formal theory survives in both its strong and weak forms. Straightforward methods of falsifying these forms have been described.

Many criticisms of the three surviving versions of matching theory can be advanced. For example, one could argue that both forms of the formal theory are incomplete because they fail to include important variables, such as reinforcer magnitude, that are known to affect response rate. It is possible to develop a more comprehensive formal theory, although no attempt will be made to do so here. Regardless of how magnitude, or any other independent

variable, might be incorporated into a more comprehensive account, the formal theory will still entail an equal- k requirement and, consequently, will be falsifiable by means of the concurrent-schedule experiment described above.

One criticism of the weak forms of matching theory is that they entail too many free parameters. Another criticism of these forms is that they can be taken to imply that empirical matching must hold in the absence of distorting phenomena, and yet this implication cannot be disconfirmed (Rachlin, 1971). A third criticism of the weak forms is that they reflect an inadequate understanding of some of the variables of which behavior is a function—namely, those that cause bias, undermatching, and overmatching (cf. Catania, 1981). The weak forms treat the effects of these variables as anomalies, but one could argue that the variables themselves should appear explicitly in an adequate theory of behavior.

These and other criticisms of the remaining versions of matching theory must be addressed should attempts to disconfirm these versions fail. In the meantime, matching remains a viable theory of behavior, the strength of which lies in its general conceptualization of all behavior as choice, and in its unified mathematical treatment of single- and multialternative environments.

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APPENDIX

Derivation of the Equations in Table 1

The three equations in the first row of Table 1 are statements of the matching principle. All of the remaining equations in the table can be produced from them. The equations in the last row are obtained immediately by letting $R_1 + R_2 + R_e = k$ in the three first-row equations. Solving each of these equations for the remaining response rate (R_1 , R_2 , or R_e) produces the three last-row equations.

The equations in rows 5 and 6 are produced as follows. Consider Equation 1-1. Dividing this equation by Equation 1-2 produces Equation 5-1. Dividing the same equation by Equation 1-3 produces Equation 6-1. Consider next Equation 1-2. Dividing this equation by Equation 1-1 produces Equation 5-2. Dividing it by Equation 1-3 produces Equation 6-2. Finally, consider Equation 1-3. Dividing this equation by Equation 1-1 produces Equation 5-3, and dividing it by Equation 1-2 produces Equation 6-3. We have now obtained all of the equations in rows 5 through 7 from the statements of the matching principle in row 1.

Equations 2-1, 3-1, and 4-1 are obtained by considering Equation 1-1, which, upon inverting and separating terms, becomes

$$\frac{R_1}{R_1} + \frac{R_2}{R_1} + \frac{R_e}{R_1} = \frac{r_1}{r_1} + \frac{r_2}{r_1} + \frac{r_e}{r_1}. \quad (\text{A1})$$

Obviously, the first term on the left equals the first term on the right. Subtracting the value of this term (viz., 1) from both sides of the equation leaves two terms on each side. Combining these terms and reinverting produces Equation 4-1. Consider Equation A1 again. We know from Equation 5-2, which has already been obtained, that the second terms on the left and right of Equation A1 are also equal. Subtracting the value of this term from both sides of the equation leaves two terms on each side. Combining these terms and reinverting produces Equation 3-1. Consider Equation A1 once more. We know from Equation 5-3 that the last terms on the left and right of Equation A1 are also equal. Subtracting the value of this term from both sides of Equation A1 leaves two terms on each side. Combining these terms and reinverting produces Equation 2-1. Equations 2-2, 3-2, and 4-2 are obtained from Equation 1-2, and Equations 2-3, 3-3, and 4-3 are obtained from Equation 1-3 in the same way.

Derivation and Discussion of Equation 11

Consider the following two weak forms, which appear in the text and were used to derive Equation 9:

$$\frac{R_e}{R_1} = \frac{1}{b_{1e}} \left(\frac{r_e}{r_1} \right)^{a_{1e}} \quad (\text{A2})$$

and

$$\frac{R_2}{R_1} = \frac{1}{b_{12}} \left(\frac{r_2}{r_1} \right)^{a_{12}}. \quad (\text{A3})$$

Consider also the following weak form, which was used to derive Equation 10:

$$\frac{R_e}{R_2} = \frac{1}{b_{2e}} \left(\frac{r_e}{r_2} \right)^{a_{2e}}. \quad (\text{A4})$$

Equation 9 is the weak form of a first-column equation in Table 1 and Equation

Appendix (Continued)

10 is the weak form of a second-column equation. An important property of the equations in Table 1 is that they are related to each other (i.e., they are dependent or interderivable) both within and between columns. Both types of dependency were illustrated in the first section of this Appendix. The between-column dependency, which is the dependency of interest in this section, is required by the matching principle. As noted in the text, the equations in the first row of Table 1 sum to the identity, $1 = 1$, as a consequence of matching. The weak forms of the first-row equations must also sum to this identity; otherwise, the matching principle is violated. The derivation of Equations 9 and 10 in the text does not introduce the required between-column dependency. This dependency may be introduced by obtaining Equation A4, which leads to the weak form of a second-column equation, from Equations A2 and A3, which lead to a weak form of a first-column equation. Dividing Equation A2 by Equation A3 produces

$$\frac{R_e}{R_2} = \frac{b_{12}}{b_{1e}} \left(\frac{r_e}{r_1} \right)^{a_{1e}} \left(\frac{r_1}{r_2} \right)^{a_{12}}. \quad (\text{A5})$$

Given the between-column dependency required by matching, the right side of this equation must equal the right side of Equation A4:

$$\frac{b_{12}}{b_{1e}} \left(\frac{r_e}{r_1} \right)^{a_{1e}} \left(\frac{r_1}{r_2} \right)^{a_{12}} = \frac{1}{b_{2e}} \left(\frac{r_e}{r_2} \right)^{a_{2e}}. \quad (\text{A6})$$

Before considering Equation A6 further, let us first verify that Equation A5 introduces the between-column dependency required by the matching principle. To do so it will be necessary to obtain weak forms, using Equation A5 instead of Equation A4, of the first-row equations in Table 1 and to show that these weak forms sum to the identity, $1 = 1$. The required weak form of Equation 1-1 is the same as that given in the text (i.e., the equation preceding Equation 9) because it does not involve Equation A5:

$$\frac{R_1}{R_1 + R_2 + R_e} = \left[\frac{1}{b_{1e}} \left(\frac{r_e}{r_1} \right)^{a_{1e}} + \frac{1}{b_{12}} \left(\frac{r_2}{r_1} \right)^{a_{12}} + 1 \right]^{-1}. \quad (\text{A7})$$

The required weak form of Equation 1-2 is obtained in the same way, except that Equation A5 is used instead of Equation A4:

$$\frac{R_2}{R_1 + R_2 + R_e} = \left[\frac{b_{12}}{b_{1e}} \left(\frac{r_e}{r_1} \right)^{a_{1e}} \left(\frac{r_1}{r_2} \right)^{a_{12}} + b_{12} \left(\frac{r_1}{r_2} \right)^{a_{12}} + 1 \right]^{-1}. \quad (\text{A8})$$

The same method, using the reciprocal of Equation A5, produces the required weak form of Equation 1-3:

$$\frac{R_e}{R_1 + R_2 + R_e} = \left[\frac{b_{1e}}{b_{12}} \left(\frac{r_1}{r_e} \right)^{a_{1e}} \left(\frac{r_2}{r_1} \right)^{a_{12}} + b_{1e} \left(\frac{r_1}{r_e} \right)^{a_{1e}} + 1 \right]^{-1}. \quad (\text{A9})$$

It must now be shown that these three equations sum to the identity, $1 = 1$. The calculations can be simplified by letting

$$A = b_{12} \left(\frac{r_1}{r_2} \right)^{a_{12}}$$

and

$$B = b_{1e} \left(\frac{r_1}{r_e} \right)^{a_{1e}}.$$

Appendix (Continued)

Making these substitutions in Equations A7 through A9, we have

$$\frac{R_1}{R_1 + R_2 + R_e} = \frac{1}{1/B + 1/A + 1},$$

$$\frac{R_2}{R_1 + R_2 + R_e} = \frac{1}{A/B + A + 1},$$

and

$$\frac{R_e}{R_1 + R_2 + R_e} = \frac{1}{B/A + B + 1}.$$

Summing these equations,

$$1 = \frac{1}{1/B + 1/A + 1} + \frac{1}{A/B + A + 1} + \frac{1}{B/A + B + 1},$$

$$1 = \frac{AB}{A + B + AB} + \frac{B}{A + AB + B} + \frac{A}{B + AB + A},$$

and combining terms

$$1 = \frac{AB + B + A}{AB + B + A},$$

$$1 = 1,$$

produces the required identity. Consequently, Equation A5 introduces the between-column dependency required by the matching principle, and the condition specified by Equation A6 must hold if matching is to be preserved. Rearranging Equation A6 produces

$$\frac{1}{b_{1e}} \left(\frac{r_e}{r_1} \right)^{a_{1e}} b_{12} \left(\frac{r_1}{r_2} \right)^{a_{12}} b_{2e} \left(\frac{r_2}{r_e} \right)^{a_{2e}} = 1, \quad (11)$$

which is the condition specified by Equation 11 in the text.