*THE EFFECT OF LOGARITHMIC TRANSFORMATION ON
ESTIMATING THE PARAMETERS OF THE
GENERALIZED MATCHING LAW*

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The generalized matching law was initially stated as a nonlinear relation between reinforcement-rate ratios and response-rate ratios. Often, the variables of the law are transformed logarithmically to remove the nonlinearity; empirical results are then fit to the model through least-squares regression. However, the logarithmic expression of the matching law is a biased statistical representation of the law itself. In particular, the logarithmic transformation alters the quantitative conclusions to be drawn from a least-squares regression analysis. A Monte Carlo study of the effect of transforming matching-law data demonstrated that (a) the estimates of one or both of the parameters of the generalized matching law are biased, (b) the measure of goodness of fit (R^2) is inaccurate, and (c) predictions generated by the fitted parameters are incorrect. Alternative approaches to logarithmic transformations are shown to alleviate these problems.

Key words: matching law, generalized matching, concurrent schedules, least-squares regression, logarithmic transformation

Studies of choice are designed to quantify the relationship between reinforcement and the distribution of behavior. Baum (1974) suggested that the results of many of these studies could be described by a generalized version of the matching law:

$$B = kR^a, \quad (1)$$

where B refers to the ratio of the rate of responding on Alternative i to the rate of responding on Alternative j , R refers to the ratio of the respective reinforcement rates, and k and a refer to parameters that describe different sources of control.

As a descriptive law, Equation 1 is used to characterize various aspects of a behavioral experiment. For example, the value of the parameter k is usually interpreted as reflecting some variable that has not been identified or controlled by the experimenter, whereas the value of a reflects the degree of sensitivity of the organism to manipulated reinforcement variables (Baum, 1974). The equation itself is used to represent the range of outcomes of a

given experiment by describing how specific values of the dependent variable (the ratio B) are functionally related to observed values of the independent variable (the ratio R).

It is rarely the case, however, that the results of an experiment can be described completely by Equation 1. Rather, uncontrolled factors and measurement errors affect the observations. Consequently, Equation 1 must be fit to empirical data through a statistical procedure, such as the method of least squares. This method provides estimates of the parameters as well as measures of the extent to which the data fit the equation.

The generalized matching law presents some problems for statistical analysis because the parameters enter into the underlying equation in a nonlinear manner. Consequently, Baum (1974) suggested that another equation be used to analyze data within the generalized matching law:

$$\log(B) = \log(k) + a \cdot \log(R) \quad (2)$$

Equation 2 is a linear expression and can be applied readily to a set of empirical observations through linear least-squares estimation. The procedure results in least-squares estimates of $\log(k)$ and a , and in a measure of the regression of $\log(B)$ on $\log(R)$.

Equation 2 has been a popular method of analysis for studies of concurrent schedules of

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reinforcement. For example, reviewers of such studies summarize outcomes in terms of estimates derived from the application of Equation 2 (e.g., Baum, 1979). When Equation 2 is used to analyze the results of an experiment, it is commonly assumed that the results of the analysis generalize to Equation 1. This assumption seems plausible because of the mathematical correspondence between the two equations.

The relation between Equation 1 and Equation 2 is an important issue, because theories of behavior usually employ Equation 1. For example, Allen (1981) and Wearden (1980) have discussed matching in connection with psychophysical power laws. Behavioral experiments, on the other hand, commonly use Equation 2. Accordingly, the theoretical motivation of the generalized matching law is based on Equation 1, whereas the empirical support is provided by Equation 2. The empirical validity of the matching law therefore depends on the correspondence between the two equations.

In this report we examine the relation between the generalized matching law (Equation 1) and the log-ratio equation used to test it (Equation 2) in three ways. First, we consider whether the least-squares fit of Equation 2 produces accurate estimates of the parameters of the generalized matching law. Second, we ask whether the least-squares fit of Equation 2 results in predictions that are the same as those of Equation 1. Finally, we examine whether the measure of goodness of fit produced by using Equation 2 adequately assesses the way the data conform to the generalized matching law.

STATISTICAL CONSEQUENCES OF THE LOGARITHMIC TRANSFORMATION

At issue is the effect of transforming the theoretical law given by Equation 1 into another equation such as Equation 2. In general, the logarithmic transformation can induce some significant distortions in the results obtained from statistical estimation. The mathematical bases for these differences have been developed by Goldberger (1968), Heien (1968), Teekens and Koerts (1972), and others. It will be helpful to review some aspects of these arguments.

The statistical problems of a transformed equation arise from the role that measurement errors play in the theory of least squares. When it is fit to empirical data, the generalized matching law (Equation 1) must be represented as a function of these errors. That is, each observation B_i is a function of the right-hand side of Equation 1 and the error associated with that observation, e_i :

$$B_i = f(k \cdot R_i^a, e_i) \quad (3)$$

The least-squares method estimates the value of e_i by the difference between the *observed* value of B_i (which contains an error component) and the *predicted* value, $\hat{B}_i$ (which does not). An important assumption therefore concerns the way error is conceptualized to affect observations and the value it is assumed to have. In the prototypical case of least squares, where X and Y are independent and dependent variables, respectively, and a and b are parameters, it is assumed that

$$Y_i = a + b \cdot X_i + e_i$$

and

$$\hat{Y}_i = a + b \cdot X_i.$$

Under this assumption, the difference between Y_i and $\hat{Y}_i$ is an estimate of e_i .

Either of two error models are commonly assumed in statistical analysis. The most common is one that assumes that error is additive. Under this model, Equation 3 becomes:

$$B_i = k \cdot R_i^a + e_i. \quad (4)$$

Clearly, the additive error complicates the transformation of the generalized matching law into its logged ratio version (Equation 2). Because the logarithm of a sum is not equal to the sum of the logarithms, the logged ratio version is not algebraically equivalent to the functional form tested by least-squares analysis (i.e., Equation 1 with additive error). In other words, the difference between $\log(B_i)$ and $\log(\hat{B}_i)$ is no longer a consistent estimator of e_i . We will provide a more concrete demonstration of this problem below.

A more tractable error model is the multiplicative model. In this form, Equation 3 becomes:

$$B_i = k \cdot R_i^a \cdot e_i. \quad (5)$$

When this expression is transformed by taking

the logarithms of both sides, the equation becomes:

$$\log(B_i) = \log(k) + a \cdot \log(R_i) + \log(e_i). \quad (6)$$

In this equation, error is represented as an additive term, which is congruent with the least-squares model.

Although the logarithmic transformation produces a linear error term, a more subtle problem remains. The problem can be understood by considering the role that errors play in the multiplicative model. Conceptually, they are assumed to provide random perturbations around some “true” value. To understand the effects of these fluctuations, one must be able to specify their distribution. That is, e_i is conceptualized as a random variable and characterized by a mean value, variance, and so on.

Over the long run, we would expect the errors in Equation 5 to cancel out. In other words, over many observations of the behavior ratio, B_i , the average B_i would be the same as a model without error (i.e., Equation 1). The consequence of this assumption is that the expected value of e_i should be 1.0, because any other value would alter the outcome of the power expression ($k \cdot R_i^a$). Furthermore, values of e_i must be nonnegative, because negative values of a behavior ratio have no meaning. An error distribution that meets these requirements is the lognormal distribution. An example of this distribution is depicted in the upper panel of Figure 1.

Like the normal distribution, the lognormal distribution is characterized by mean and variance parameters. Indeed, there is an exact relationship between the lognormal and normal distributions, in that the logarithm of a lognormal distribution is a normal random variable. This means that the error term of Equation 6, $\log(e_i)$, is normally distributed. This relationship can be seen by comparing the two panels of Figure 1. The lower panel depicts the natural logarithmic transformation of the values in the upper panel. As can be seen, the distribution has a normal shape. Consequently, the normality assumption of least-squares analysis is tenable when the errors have been transformed.

The lognormal distribution of error implies that unknown factors are operating in an interactive manner. The interaction of reinforcement variables is a common assumption of extensions of the matching law (de Villiers, 1977).

Fig. 1. Upper panel: A frequency histogram of 49,500 random numbers distributed according to a lognormal distribution. The numbers were generated according to the algorithm described in the text. Lower panel: A frequency histogram of the natural logarithmic transformation of the numbers depicted in the upper panel.

Accordingly, we will assume e_i in Equation 5 to be lognormal with expectation of 1 and variance ω^2 . Under this assumption, the error term of Equation 6, $\log(e_i)$, is normally distributed. The expectation, or mean, of the logarithm of a lognormal random variable can be readily derived (e.g., Teekens & Koerts, 1972). This derivation shows that when the expectation of the lognormal random variable is 1.0, as we have assumed above, the expectation of its logarithm is not zero—a fact confirmed by comparison of the means in Figure 1. In the transformed distribution (lower panel), the mean is a function of the variance. Consequently, although the log transform of the multiplicative model has the distribution required by linear least-squares methods, the expectation of the error term is not zero, as required. The result of this violation is that the estimate of the intercept coefficient is biased. Hence, the estimate of $\log(k)$ will not be the true value.

Both additive (Equation 4) and multiplicative (Equation 5) models exhibit an additional difficulty because of these mathematical relations. The estimated parameters are, of course, those specified by the generalized

matching law (Equation 1). Ultimately, behavioral researchers are interested in k rather than $\log(k)$, because the former is used to estimate and predict the actual dependent variable B . For example, a behavior analyst attempting to control a client's compulsive gambling would be unlikely to discuss it in terms of logarithms (i.e., "Change log ratio reinforcement to increase log ratio savings."). The least-squares fit of the transformed equations is a procedure that estimates $\log(k)$. Using the antilog of this estimate to predict the dependent variable B_i will produce transformation bias (Miller, 1984) in the prediction. It follows that the behavioral predictions of the generalized matching law will be inaccurate (Teekens & Koerts, 1972).

Furthermore, measures of goodness of fit are based upon the predicted values of the dependent variable. If these values differ between Equation 1 and Equation 2, then a goodness-of-fit measure, such as R^2 may not be the same under both equations (Kvålseth, 1983, 1985).

MONTE CARLO SIMULATIONS

These problems can be illustrated more concretely. Suppose a certain behavioral condition resulted in matching according to Equation 1, with known and constant values of k and a . Measurement error will obscure the functional relationship so that, for any one experiment, the best empirical estimates of k and a differ from the true values. It is reasonable to expect that a large number of experiments with this condition would provide more accurate estimates. In this section we examine whether this convergence happens if Equation 2 is used to estimate data that conform to Equation 1. Specifically, over a large number of experiments, do the estimates of k and a correspond to their true values? Do predictions made under Equation 1 correspond to values computed using the fitted parameters of Equation 2? Finally, does the use of Equation 2 lead to accurate measures of R^2 ?

To answer these questions, we performed a series of Monte Carlo simulations of matching according to Equation 1. Eleven values of the ratio R were chosen. Each of these values was substituted into Equation 1, and the expression evaluated using set values of k and a . The result of this substitution was a series of 11 simulated values of the behavior ratio. Error terms were

then randomly generated for each behavior ratio. These random disturbances were used to compute observed values of the behavior ratios, reflecting the influence of measurement error. The distribution of the error terms, and the way they were combined with the Equation 1 behavior ratios, depended on the error model. All calculations were performed with a Macintosh® computer using Trapeze®, a spreadsheet program.

After the 11 randomly displaced behavior ratios were computed, the natural logarithms of both reinforcement ratios and behavior ratios were used in a linear regression analysis. This analysis produced estimates of $\log(k)$ and a for that set of experimental values. This procedure was repeated to give 500 replications of the Monte Carlo experiment. Each replication will be referred to as a trial. At issue is whether the results of the 500 trials yield values of a and k that are the same as specified by the values in the model (i.e., the true values).

This procedure requires a specification of the error model, as well as the actual distribution of error values. Accordingly, we now consider the two error models discussed above—the additive model of Equation 4 and the multiplicative model of Equation 5.

The additive model can be used only over a restricted set of values. Because the sum of the error term and $k \cdot R_i^a$ in the generalized matching equation (Equation 1) is limited to positive values, the error term cannot have large negative values. Negative values of B_i would not only be uninterpretable in behavioral terms, but would also preclude logarithmic transformation. For example, consider Equation 4 when a is 1, k is 1, and two values of R are used, 1.0 and 0.1. An error term of -0.15 would result in measured values of B_i equal to 0.85 and -0.05 , respectively. Of course, a negative behavior ratio has no meaning.

An additive error model is possible only if the experiment is restricted to large values of predicted B . Such a restriction limits the utility of the additive model; therefore, we will provide only one test of this model under appropriate conditions. For this test, we chose values of a equal to 0.8 and k equal to 1.67; R was tested at values from the series 0.40, 0.56, 0.80, 1.13, 1.60, 2.26, 3.20, 4.25, 6.40, 9.05, and 12.80. The parameters a and k were set so as to reflect a condition of undermatching and

strong bias; the values of R generate logarithms that are equally spaced and symmetric about $\log(2.26)$.

The errors were generated by selecting random values over the interval 0 to 1.0, provided by the Trapeze® software. The random values, u_i , are uniformly distributed over this interval. They were then substituted into the formula $e_i = (u_i - 0.5)/0.645$, where the constants 0.5 and 0.645 were used to adjust the values to have a mean of 0 and a variance of 0.2, over a rectangular distribution. The rectangular distribution and these constants were chosen so that values of e_i would be limited to the range -0.78 to 0.78 , avoiding negative values of B_i .

For each of the 500 trials, the natural logarithms of the displaced values were fit to the natural logarithms of the 11 reinforcement ratios. The results of these analyses are presented in Figure 2. This figure has three panels depicting the cumulative frequency curves for statistics derived from the Monte Carlo trials. The cumulative frequency curves provide a graphical estimate of the underlying distribution. If the distribution were Gaussian, then these curves should be ogival, or S-shaped and symmetric about the mean.

The upper and middle panels of Figure 2 depict the distribution of the 500 estimates of k and a , respectively. For these parameters, the mean estimates were 1.55 and 0.86. The vertical lines in Figure 2 indicate the true values of k and a used to test the additive model. The horizontal lines depict the 50th percentile; these lines intersect the cumulative frequency curves at the median value. The medians were 1.55 and 0.83 for the values of k and a , respectively. The steepest segment of the cumulative frequency curve, which is an estimate of the mode, occurred in the interval between 1.70 and 1.75 for k , and in the interval between 0.78 and 0.80 for a . For both parameters, then, the average outcome of the 500 Monte Carlo experiments does not equal the true value of the parameters. The separation of the mode from the other two measures of central tendency indicates that the distribution of the obtained outcomes is not Gaussian.

To provide another measure of the adequacy of the regression analysis, the estimated values of the parameters were used to predict a value for a reinforcement ratio of 15.0. The predicted value was calculated using Equation 1 for each

Fig. 2. The results of log ratio regression analysis of Monte Carlo simulation of the additive model. There were 500 trials, each yielding estimates of k (the natural antilog of the intercept term of the linear regression), a , and the predicted value of B when $R = 15.0$. The upper panel depicts the cumulative frequency histogram of the k estimates, the middle panel depicts the cumulative frequency of histogram of a , and the bottom panel depicts the cumulative frequency histogram of predicted B . The vertical line in each panel marks the "true" value of the parameter used to generate the Monte Carlo data. The horizontal line marks the 50th percentile. The median can be determined by dropping a perpendicular from the intersection of this line with the curve to the x axis.

of the 500 trials; the cumulative frequency plot of these predicted B values is given in the lower panel of Figure 2. The mean value was 15.60. The true value of B , when k is 1.67 and a is 0.8, is indicated by the vertical line. The observed median B can be determined from the horizontal line and was 14.96, and the mode occurred in the interval between 14.75 and 15.00. In this simulation, the average B values overestimate the true B value of 14.57.

The Monte Carlo analysis also provided two estimates of the degree of fit. One estimate, Equation 2 R^2 , is the proportion of variance of the transformed dependent measures ac-

counted for by the linear relationship with the transformed independent measures. The other, Equation 1 R^2 , is the proportion of variance of the untransformed dependent measures accounted for by the untransformed independent measures, estimated by using Equation 1. The computation of Equation 1 R^2 used the value of the slope coefficient from the linear fit as the value of the parameter a and the natural antilog of the intercept coefficient as the value of the parameter k . For Equation 2 R^2 the mean value was 0.93, with a median of 0.94, and a mode in the interval between 0.94 and 0.95. The results for Equation 1 R^2 were 0.96 for the mean, 0.98 for the median, and an interval of 0.98 to 0.99 for the mode. These measures show that the value of R^2 obtained from the log-ratio analysis (Equation 2 R^2) does not correspond to the way Equation 1 describes the data (Equation 1 R^2). Kvålseth (1985) has demonstrated that the latter value should be used to represent the fit of an equation. In contrast, behavior analysts have usually reported R^2 based on Equation 2. The discrepancy is not large in the present case, but we will show that substantial differences can occur under other error models.

It seems reasonable to attribute the error in estimates of k , a , B , and R^2 to the distortions induced in the additive model by the logarithmic transformation. Because the transformation is applied to a sum, the resulting value is not congruent with the form of the linear equation fit by the linear least-squares procedure. That is, given the error structure of the additive model, the linear least-squares approach attempts to minimize the squared difference between $\log(\hat{B}_i)$ and $\log(B_i + e_i)$. The result is not equal to e_i^2 .

The multiplicative model is more justifiable than the additive model because it results in linearity of the error term under transformation. In addition, by assuming a lognormal error distribution, the multiplicative model has greater generality than does the additive. Because the error terms must be positive, the product of the error term and values of R will result in values of B that are positive throughout the range of R . Finally, Tustin and Davison (1978) reported that the residuals of a regression analysis of Equation 2 were normally distributed. This outcome is consistent with a lognormal distribution of error for the untransformed variables.

To test the effect of logarithmic transformation on the multiplicative model, several Monte Carlo simulations were conducted using a factorial combination of different k and a values. For each combination, 500 trials were conducted using Equation 5 to generate the individual B_i observations. Using the random number function of Trapeze®, uniform random numbers were first converted into standard Gaussian random variates, using a Box-Muller procedure (Press, Flannery, Teukolsky, & Vetterling, 1986):

$$X = \sqrt{-2 \ln(u_1)} \cos(2\pi u_2)$$

where X is the random normal variate, and u_1 and u_2 are two random uniform variates in the range of 0 to 1. The standard normal variates resulting from this procedure have a mean of 0 and a standard deviation of 1. They were then converted by an inverse z -score transformation to a Gaussian distribution with a mean of -0.91 and a variance of 0.182 . These values were then raised to the power e , the natural exponent. The result of the Box-Muller procedure, the z -score transformation, and the exponentiation procedure, is a lognormal variate with a mean of 1 and a variance of 0.2. These values of the random variable were multiplied by the term $k \cdot R_i^a$ to yield observed values of B_i .

Three values of k were tested—0.6, 1.0, and 1.67, corresponding to negative, zero, and positive bias for the alternative whose reinforcement rate appears in the numerator of the matching ratio. Three values of a were chosen—0.8, 1.0, and 1.2, which correspond to undermatching, matching, and overmatching, respectively. These values produced nine treatment combinations for simulation. For each combination, R_i values of 0.03125, 0.0625, 0.125, 0.25, 0.5, 1.0, 2.0, 4.0, 8.0, 16.0, and 32.0 were employed, which resulted in equally spaced logarithmic values symmetric about zero. The simulation therefore employed R_i values similar to those that might be used in an actual experiment. As in the case of the additive analysis, the observed values of B_i and R_i were converted to logarithms and entered into a linear regression. Each simulation trial therefore resulted in estimated values of k , a , predicted B for $R = 15$, and the two measures of R^2 .

The results of these simulations are presented in a series of figures that graph the

Fig. 3. Estimates of a generated by the log ratio regression analysis of the Monte Carlo simulations of the multiplicative model. In the upper panel, the value of k used in the simulations was 0.6. In the middle panel, $k = 1.0$. In the lower panel, $k = 1.67$. The different curves depict the cumulative frequency of estimates produced when the value of a used in the simulations was equal to 0.8, 1.0, and 1.2. Vertical lines mark these *a priori* values of a ; horizontal lines indicate the 50th percentile. Because the horizontal line intersects the curve near the vertical line, the median is close to the true value.

cumulative frequency curves for the 500 trials. Figure 3 depicts the effects of the regression estimate of a for different combinations of the a and k parameters. The distributions indicate that the average value of the a estimates is virtually identical to the true value. Furthermore, the cumulative frequency curves appear close to the ogival shape, indicating that these estimates are normally distributed. Thus, when a multiplicative model with lognormal error is assumed, the use of a logarithmic transformation, as in Equation 2, does not distort the estimate of a .

This conclusion does not hold in the case of the average estimate of k . Figure 4 presents the results of the nine simulation tests with

Fig. 4. Estimates of k (the natural antilog of the intercept term) generated by the log ratio regression analysis of the Monte Carlo simulations of the multiplicative model. In the upper panel, the value of k used in the simulations was 0.6. In the middle panel, $k = 1.0$. In the lower panel, $k = 1.67$. The different curves depict the cumulative frequency of estimates produced when $a = 0.8, 1.0$, and 1.2 . Vertical lines mark the true values of k ; horizontal lines indicate the 50th percentile. The horizontal line intersects the curve at the median value; note the distance of this point from the true value.

respect to the k estimates. Several features are apparent in Figure 4. Note that the estimates are not affected by variation in a . For every curve, the median line intersects the curves at a point below the true value of k . Also, the steepest part of the curve, reflecting the mode, lies below the true value. This underestimation is also apparent in the mean values for each curve. When k was set at 0.6, the mean estimated value was 0.56, 0.55, and 0.56 for a values of 0.8, 1.0, and 1.2, respectively. When k was set to 1, these three values were 0.92, 0.93, and 0.93, and when k was set to 1.67, the values were 1.53, 1.54, and 1.55. Thus, the results clearly indicate that k is underestimated when Equation 2 is used to provide an em-

Fig. 5. Estimates of B generated by the log ratio regression analysis of the Monte Carlo simulations of the multiplicative model. B was computed by using the estimates of k and a produced by the analysis in Equation 1, with R equal to 15.0. In the upper panel, the value of k used in the simulations was set to 0.6. In the middle panel, $k = 1.0$. In the lower panel, $k = 1.67$. The different curves depict the cumulative frequency of estimates produced when $a = 0.8, 1.0$, and 1.2 . Vertical lines mark the true values of B ; horizontal lines indicate the 50th percentile. The horizontal line intersects the curve at the median value; note the distance of this point from the true value.

pirical test of Equation 1. The amount of underestimation is not affected by the value of a . Note also that underestimation appears even when the true value of k is 1.0.

This underestimation is also apparent when the derived values of k and a are used to predict a value for B from a given value of R . Figure 5 displays the results of these predictions for the different combinations of k and a when R is set to 15. The true values are indicated by the vertical lines. It can be seen that the median values are below the true values. When k was set to 0.6, deviations from true values (the mean predicted values of B minus the true values given by Equation 1) were -0.35 , -0.65 , and -0.83 for a equal to 0.8, 1.0, and 1.2, respectively. When k was set to 1.0, de-

Fig. 6. Computed values of R^2 generated by the log ratio regression analysis of the Monte Carlo simulations of the multiplicative model. As described in the text, R^2 was computed by two methods. The three curves on the right of each panel depict the cumulative frequency of R^2 when it is computed as the square of the linear regression of log ratio values (described in the text as Equation 2 R^2). The three curves on the left of each panel depict R^2 when the estimates produced by the regression analysis are used in Equation 1 to predict a specific value of B , for computing the sum of squares of deviations of observed from predicted (described in the text as Equation 1 R^2). In the upper panel, the value of k used in the simulations was set to 0.6. In the middle panel, $k = 1.0$ and in the lower panel, $k = 1.67$. The different curves depict the cumulative frequency of R^2 values produced when $a = 0.8, 1.0$, and 1.2 . Horizontal lines indicate the 50th percentile.

viations were -0.60 , -0.93 , and -1.56 . When k was equal to 1.67, deviations were -0.98 , -1.56 , and -2.69 .

As noted in the case of the additive model, there are two methods of calculating R^2 . One method is to square the correlation coefficient calculated from the regression analysis of Equation 2. The cumulative distribution of these R^2 values (Equation 2 R^2) appears at the right of each panel in Figure 6. It can be noted that these values generally exceed 0.90. The means ranged from 0.95 to 0.98. Alter-

natively, R^2 can be calculated as the proportion of variance in the observed values of B accounted for by Equation 1 (Equation 1 R^2), depicted by the curves to the left in Figure 6. Surprisingly, the distribution of these R^2 values is markedly different, with most of the values falling below 0.90. Across the different Monte Carlo simulations, the mean values ranged from 0.83 to 0.85. The results indicate that R^2 based on Equation 2 overestimates the accuracy of the matching law as expressed in Equation 1.

To summarize, the Monte Carlo study demonstrated that the log-ratio analysis is an inadequate method to fit the multiplicative model. The log-ratio analysis provides an estimate of R^2 that might misleadingly suggest an acceptable level of fit between data and equation. The true fit, however, is not as good. Furthermore, the estimate of the parameter k is inaccurate, leading to predictions for new values that are discrepant from the general model. Such inaccuracy would be important to models that ascribe special significance to the k parameter. For example, Houston and McNamara (1981) have developed an optimality model that implies a general form of Equation 1, but within which the equivalent of the k parameter is a function of the optimal allocation of behavior and of the changeover delay. For this model, the numerical value of k is a means of testing the model empirically.

Because the multiplicative model does preserve the validity of the log transformation, it permits an analytical description of the magnitude of the discrepancy. Teekens and Koerts (1972) have shown that for both simple and multiple regression one would expect

$$k - k_{est} = k[1 - \exp(-.5\sigma^2 + .5\sigma_1^2)] \quad (7)$$

where k is the true value, k_{est} is the value estimated from the log-ratio approach, σ^2 is the variance of the transformed error term and σ_1^2 is the variance of the intercept term. The obtained value from the Monte Carlo simulations can be compared to this theoretical value. When k was equal to 0.6, 1.0, and 1.67, the values of k_{est} expected under the multiplicative model were 0.55, 0.92, and 1.54, respectively, which agree with the obtained values from the Monte Carlo trials.

Equation 7 can be used to extend the analysis of the multiplicative model to situations with different error variances. For example, if the variance of the error term is 0.4, then σ^2

= 0.34. With the values of R used in the Monte Carlo simulations, $\sigma_1^2 = 0.03$, and the estimates of k would be 86% of their true values.

It should be noted that the error in estimation described by Equation 7 applies to the case of a multiplicative model in which the values of the independent variable are measured without error. It may be that the reinforcement ratios themselves are subject to uncontrolled variation: Mechanical factors might affect the accessibility of a grain magazine, rat food pellets might crumble and be lost and so on. Error in the measurement of R complicates the regression analysis of the matching law (Davison & McCarthy, 1988). A new model must be specified and more powerful regression techniques must be employed (e.g., Lybanon, 1984). Monte Carlo analyses suggest that when the log-ratio procedure is used to estimate the multiplicative model, the underestimation of k is reversed, a is underestimated, and the difference between Equation 1 R^2 and Equation 2 R^2 is large. The effects vary with the values of the parameters and the variance of error in the independent and dependent variables, and can be quite large. In one simulation, using error variances of 0.3 and 0.2 for independent and dependent variables, respectively, we found a mean Equation 2 R^2 of 0.94 and a mean Equation 1 R^2 of 0.60.

The use of the log-ratio approach to estimate the parameters of the generalized matching law can therefore induce systematic biases. The extent of these biases is difficult to gauge. It can also be noted that, because the amount of bias is dependent upon the variance in the error term, different conditions within a study could reflect different amounts of bias. In such a case, the ordinal relations among k estimates would not necessarily hold. By emphasizing the experimental reduction of error rather than statistical description, the experimental analysis of behavior undoubtedly mitigates some of these problems. Nevertheless, it seems advisable to provide some means of evaluating and correcting them. Two approaches are considered in the next section.

ALTERNATIVES TO THE LOG-RATIO METHOD

One reason for the popularity of the log-ratio method of analysis is that the estimates produced in estimating Equation 2 have sim-

ple graphical representations. Specifically, when $\log(B)$ is plotted against $\log(R)$, the value of the parameter a gives the slope of the best-fitting line and the value of $\log(k)$ gives the intercept with the y axis. It should be recognized that this apparent simplicity is achieved at the cost of a more complex scale. Nevertheless, the use of this coordinate system has had a long tradition, and many investigators may wish to preserve it. Consequently, one remedy to the bias produced by the log-ratio analysis is to consider ways in which the analysis could be retained.

We suggest as one approach that investigators consider a detailed specification of the error model. Leech (1975), for example, has presented a means for testing whether the error term is additive or multiplicative. If the latter is tenable, a log-ratio analysis could be considered provided investigators report the mean square error of their analyses. This statistic can then be used to calculate the amount of bias in k (Miller, 1984) or to compute new estimators for the prediction of B (Teekens & Koerts, 1972). Investigators should also consider the appropriate measure of R^2 . If the purpose of the experiment is the description or prediction of the functional relation $B = f(R)$, then the measure of the percentage of variance accounted for should be calculated using the values of variables and parameters after the transformation has been reversed (Kvålseth, 1985).

Alternatively, the generalized matching law (Equation 1) could be fit using nonlinear methods. Mainframe statistical packages such as BMDP (Dixon, 1981) and some microcomputer ones such as SYSTAT (Wilkinson, 1986) provide nonlinear options. These can be used to generate least-squares estimates of the parameters of Equation 1. Within certain assumptions, as the number of observations increases, such estimates become asymptotically unbiased (Gallant, 1987).

To compare the nonlinear method to the log-ratio approach, we reanalyzed the data of the Monte Carlo trials, using Equation 1 directly. A program was written in Trapeze® to apply a Gauss-Newton solution to the nonlinear least-squares expression of Equation 1 (Jennrich & Ralston, 1979). The program iteratively adjusted the parameters k and a by small steps, $\Delta\theta$, and used the new values of k and a to compute the residual sum of squares. The

step size on Iteration i was determined by the equation

$$\Delta\theta = (\partial f / \partial \theta^T \partial f / \partial \theta)^{-1} \partial f / \partial \theta^T [y - f(\theta)] \quad (8)$$

where θ is the parameter vector, $f(\theta)$ is the value vector of Equation 1 for the values of R at θ , y is the vector of observed values of B , and $\partial f / \partial \theta$ is the Jacobian of $f(\theta)$ at θ . (A description and listing of this program have been placed in the APPVEND forum on the CompuServe computer network, in data library 5 under the name *nonlin.tpz*. The program can be downloaded through the CompuServe network. Running the program requires the Trapeze® spreadsheet package.)

The Gauss-Newton procedure is well suited to the additive model. When the 500 Monte Carlo trials were reanalyzed with this procedure, the mean value of the 500 estimates of k was 1.67; the mean estimate of a was 0.80. The mean R^2 was .99, and the mean estimate of B when $R = 15$ was 14.58. These values indicate good agreement with the true values from the Monte Carlo simulation of the additive model.

The multiplicative model was reanalyzed using the combination $k = 1.67$ and $a = 0.8$. However, a nonlinear regression using the multiplicative model requires special methods. The model is inherently heteroscedastic, due to the multiplicative error term. With the number of R values used in this analysis (11), the use of Equation 8 produces inaccurate estimates of the parameters of Equation 1. For example, the mean value of k estimated from the 500 Monte Carlo trials was 1.80; the mean value of a was 0.83.

Gallant (1987) and Jennrich and Ralston (1979) suggest modifications in nonlinear techniques to compensate for heteroscedastic error. For example, if the variances of the observations at each level of the independent variable are known, then a weighted least-squares method can be employed with the Gauss-Newton procedure. Specifically, a diagonal matrix, W , is formed, using the reciprocals of the variances as the elements along the diagonal. Equation 8 is then replaced by

$$\Delta\theta = (\partial f / \partial \theta^T W \partial f / \partial \theta)^{-1} \partial f / \partial \theta^T W [y - f(\theta)]. \quad (9)$$

To illustrate the improvement in estimation, the weighted Gauss-Newton method was reapplied to the case of $k = 1.67$ and $a = 0.8$;

the variances from the 500 Monte Carlo trials were used to construct the weight matrix. This modified procedure resulted in a mean estimated value of k equal to 1.64 and a mean estimated value of a of 0.80. The mean R^2 was 0.84, and the mean predicted value of B when R was set to 15.0 was 14.66.

In practice, it is unlikely that a researcher would have estimates of the variances available to construct the matrix $\mathbf{W}$. Under the multiplicative model, the square of the values of the independent variables might be employed as a first approximation. The techniques recommended by Gallant (1987) could also be used.

CONCLUSION

Historically, the experimental analysis of behavior eschewed techniques of analysis that obscured the fine-grained detail of its data. The use of quantitative models to describe some aspects of behavior has, to some extent, necessitated the use of statistical procedures such as least-squares regression. Such techniques should be chosen to the extent that they elucidate important details of data. The generalized matching law (Equation 1) developed by Baum (1974) is a model that specifies important distinctions between different sources of behavioral control. Equation 2 is not a statistically neutral representation of the law and can obscure some of these distinctions.

In this paper, Equation 1 is assumed to be the appropriate representation of the generalized matching law. It is interesting to note that most researchers begin with the power-law version of matching and proceed to employ the log-ratio equation as an empirical test. This progression assumes that the two equations represent the same behavioral law. As we have shown, the log-ratio version of matching may misrepresent the relations implied by the power law. The discrepancy begs the question: What is the generalized matching law?

We argue that a natural law is best stated in terms of the measurement procedures used to assess values of the independent and dependent variables. The experimental analysis of behavior has typically remained "close to the data." This is because theoretical advances occur when theory is closely tied to the techniques of measurement (e.g., Skinner, 1959). Most experiments involve simple measures based on counts or time that indicate the oc-

currence of responses or reinforcers. Because our experiments deal with these direct measures, it seems appropriate to describe them mathematically at this level. One implication is that our equations should predict the ratio of responses or time allocation rather than the logarithm of these measures. The power-law version of matching relates behavior to reinforcement at the level of measurement, and in this sense is a fundamental expression of matching. Other nonlinear equations may also be useful as representations of this law. For example, it may be possible to represent generalized matching in terms of proportions rather than ratios.

Generally, researchers should be aware of the consequences of mathematical, as opposed to statistical, operations. A model expressed as a power function can, of course, be stated as a log-linear equation. This is simply a mathematical translation. This translation, however, ignores the fact that the underlying terms of the model are random variables when the equation is fit to measured results. The statistical consequences of the translation can produce transformation bias in the parameter estimates and inaccurate descriptions of the true fit of the model. The matching law has generated a large number of studies that were evaluated in terms of the log-linear expression; many of these studies could be reassessed in terms of the power law. Such a reassessment seems necessary to depict accurately the current status of the matching law.

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