V_3 IN R_6 OF $[1,2]_a$

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I. *Introduction.*—For an introduction to our methods, we refer the reader to the paper " V_3 in R_6 ."¹

II. V_3 in R_6 of $[1,2]$.—Following the classification given in the above paper, these V_3 in R_6 are characterized by:

$$V_a = 0 \quad (1.1)$$

$$h_{ab} = h_a i_b \quad (1.2)$$

$$h_{ab} = h_a i_b \quad (1.3)$$

$$h_{ab} = h_a i_b + h_a i_b \quad (1.4)$$

Here i_1, i_2, i_3, i_7 are arbitrary but i_3 is perpendicular to i_7 . The equations of Gauss, Codazzi and Ricci are

$$K_{abcd} = \Sigma (h_{ac} h_{bd} - h_{ad} h_{bc}) \quad (2.1)$$

$$h_{a[b,c]} = \Sigma V_{[b} h_{c]a} \quad (2.2)$$

$$V_{[a,b]} = h_{[a} h_{b]} - \Sigma V_{[a} V_{b]} \quad (2.3)$$

Using (I), from the Ricci Relations (2.3), we find that V_3 of $[1,2]$ contains (4) subcases according as

$$i_1 = i_2; i_1 \text{ is perpendicular to } i_3, i_7 \quad (3.1)$$

$$i_1 = i_2; i_1, i_3, i_7 \text{ lie in the same } R_2 \quad (3.2)$$

$$i_1 \perp i_2; i_1, i_3, i_7 \text{ lie in the same } R_2 \quad (3.3)$$

$$i_1 \perp i_2; i_1, i_3 \text{ lie in the same } R_2; i_7 \perp i_2 \quad (3.4)$$

We denote these classes by $[1,2]_a$; $[1,2]_b$; etc., according as to whether (1), (3.1); (1), (3.2); etc., are satisfied. In this paper, we shall study $[1,2]_a$.

III. V_3 in R_6 of $[1,2]_a$.—The V_3 in R_6 of this class possess the following quantities:

$$V_a = 0 \quad (1.1)$$

$$h_{ab} = h_{a \atop 4} i_{b \atop 4 \atop 1 \atop 1} \quad (1.2)$$

$$h_{ab} = h_{a \atop 5} i_{b \atop 5 \atop 1 \atop 1} \quad (1.3)$$

$$h_{ab} = h_{a \atop 6} i_{b \atop 6 \atop 2 \atop 3} + h_{a \atop 7} i_{b \atop 7 \atop 3 \atop 3} \quad (1.4)$$

$$i_{\atop 1}, i_{\atop 2}, i_{\atop 3} \text{ are mutually orthogonal.} \quad (1.5)$$

For normal curvature properties, we find

(2.1) the normal curvature vectors lie in an R_2 through each point P .

(2.2) the asymptotic directions have no special significance.

(2.3) to each direction i in the R_2 perpendicular to $i_{\atop 1}$, there corresponds a plane of conjugate directions.

From the Codazzi Relations, we find

$$i_{\atop 1}, i_{\atop 2}, i_{\atop 3} \text{ are triply orthogonal } V_2 \text{ building} \quad (3.1)$$

$$V_2 \text{ of } (i_{\atop 2}, i_{\atop 3}) \text{ is geodesic in } V_3 \quad (3.2)$$

$$i_{\atop 1} \text{ is a geodesic of } V_3 \quad (3.3)$$

$$\text{using a coordinate system along } i_{\atop 1}, i_{\atop 2}, i_{\atop 3} \quad (3.4)$$

$$h = h(u'), h = h(u') \quad (3.5)$$

$$\frac{\partial h}{\partial u^2} = \frac{h - h}{h_2} \frac{\partial h_2}{\partial u^3}, \frac{\partial h}{\partial u^2} = \frac{h - h}{h_3} \frac{\partial h_3}{\partial u^2} \quad (3.6)$$

$$ds^2 = (du')^2 + (h_2 du^2)^2 + (h_3 du^3)^2 \quad (3.7)$$

$$h_2 = h_2(u^2, u^3); h_3 = h_3(u^2, u^3).$$

For the Gauss Relation, Ricci Tensor and K of V_3 , we find from (1.1)

$$\frac{\partial}{\partial u^2} \left(\frac{1}{h_2} \frac{\partial h_3}{\partial u^3} \right) + \frac{\partial}{\partial u^3} \left(\frac{1}{h_3} \frac{\partial h_2}{\partial u^2} \right) = -\frac{hh}{h_2 h_3} \quad (4.1)$$

$$K_{ad} = -\frac{hh}{h_2 h_3} (i_{a \atop 6} i_{d \atop 6 \atop 2 \atop 2} + i_{a \atop 7} i_{d \atop 7 \atop 3 \atop 3}) \quad (4.2)$$

$$K = -\frac{2hh}{h_2 h_3} \quad (4.3)$$

Since the $\infty' V_2$ of $(i_{\atop 2}, i_{\atop 3})$ are geodesic in V_3 , denoting the fundamental quantities of these $\infty' V_2$ by $(-)$, we find from the Gauss Relations for V_2 in V_3

$$\bar{K}_{232} = K_{232} \quad (5.1)$$

$$\bar{K}_{ab} = K_{ab} \quad (a, b = 2, 3) \quad (5.2)$$

$$\bar{K} = K. \quad (5.3)$$

Hence (4.1) is the Gauss Relation for the $\infty' V_2(i, i)$ in V_3 . From a simple analysis of the Weingarten Formulae, denoting any V_2 of (i, i) by $x^\lambda(u^2, u^3, c)$ and V_3 by $y^\lambda(u^1, u^2, u^3)$, we find

$$\left(\frac{\partial^2 y^\lambda}{\partial u^a \partial u^b} \right)_{u^1=c} = \frac{\partial^2 x^\lambda}{\partial u^a \partial u^b} \quad (a, b = 2, 3) \quad (6.1)$$

$$h_{ab}|_{u^1=c} = \bar{h}_{ab} \quad (p = 4, 5, 6) \quad (6.2)$$

$$i_1^\lambda = \frac{\partial y^\lambda}{\partial u^1} \quad (6.3)$$

$$\Gamma'_{ab} = \bar{h}_{ab} = 0. \quad (6.4)$$

Hence (4.1) is also the Gauss Relation for V_2 in R_6 . Finally, a simple analysis shows that the $\infty' V_2$ of (i, i) have the following fundamental quantities.

$\bar{g}_{ab}$ of a particular V_2 are arbitrary; the $\infty' V_2$ are applicable (7.1)

$$\bar{h}_{ab} = 0 \quad (p = 1, 4, 5) \quad (a, b = 2, 3) \quad (7.2)$$

$$\bar{h}_{ab} = h_{a2}i_b + h_{a3}i_b \quad (7.3)$$

$$\bar{V}_a = 0 \quad (p, q = 4, 5, 6, 1). \quad (7.4)$$

Furthermore a simple calculation shows that (3.6) are the Codazzi Relations of these $\infty' V_2$ in R_6 . Hence, the $\infty' V_2$ in R_6 are completely specified by the above.

From (7.2), (7.3) by the use of a very simple theorem, we find that the $\infty' V_2$ lie in $\infty' R_3$.

Hence the V_3 of $[1, 2]_a$ are generated by ∞' identical V_2 lying in $\infty' R_3$. The direction of generation i is orthogonal to $\infty' R_3$.

From the above geometry, one sees that the integral equations of these V_3 are of the type

$$y^\lambda = a_b^\lambda(u^1) \times^b(u^2, u^3) + w^\lambda(u^1) \quad (8.1)$$

$$(\lambda = 1, 2, \dots, 6) \quad (b = 1, 2, 3)$$

¹ Proc. Nat. Acad. Sci., 20, 489-494 (1934).