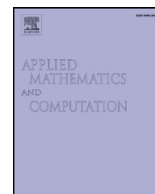




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# Effects of limited medical resource on a Filippov infectious disease model induced by selection pressure



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## ABSTRACT

In reality, the outbreak of emerging infectious diseases including SARS, A/H1N1 and Ebola are accompanied by the common cold and flu. The selective treatment measure for mitigating and controlling the emerging infectious diseases should be implemented due to limited medical resources. However, how to determine the threshold infected cases and when to implement the selective treatment tactics are crucial for disease control. To address this, we derive a non-smooth Filippov system induced by selective treatment measure. The dynamic behaviors of two subsystems have been discussed completely, and the existence conditions for sliding segment, sliding mode dynamics and different types of equilibria such as regular equilibrium, pseudo-equilibrium, boundary equilibrium and tangent point have been provided. Further, numerical sliding bifurcation analyses show that the proposed Filippov system has rich sliding bifurcations. Especially, the most interesting results are those for the fixed parameter set as the bifurcation parameter varies, the sliding bifurcations occur sequentially: crossing  $\rightarrow$  buckling  $\rightarrow$  real/virtual equilibrium  $\rightarrow$  buckling  $\rightarrow$  crossing. The key factors which affect the selective treatment measures and the threshold value of infected cases for emerging infectious disease have been discussed in more detail.

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## 1. Introduction

In the last few years, frequent outbreaks and quick spread of the emerging infectious disease become a worldwide public healthy problem. It endangers not only people's health but also the stability of the whole society. In 2003, SARS (Severe Acute Respiratory Syndrome) killed 774 people, infected more than 8000 globally and threatened to spread around the world [1,2]. The A/H1N1 (Hemagglutinin 1, Neuraminidase 1) influenza virus, which caused the 2009 pandemic, continues to circulate in some parts of the world, causing variable levels of disease and outbreaks [3]. By September 14, 2014, a total of 4507 probable and confirmed cases, including 2296 deaths from Ebola virus disease (EVD) had been reported from five countries in West Africa-Guinea, Liberia, Nigeria, Senegal, and Sierra Leone [4]. In the early stages of the emerging infectious disease, it is deficient to recognize the emergence of these infectious systematically and comprehensively, the measures of

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disinfection and isolation are failed to protect and control infected people, so the patients have little resistance to infection of any kind and emerging infectious spread through the camps like wildfire.

The emerging infectious disease including SARS, A/H1N1, Ebola, Dengue fever [5] etc. are often accompanied by other viral diseases such as common flu. Moreover, in the early stages of emerging infectious diseases (taking SARS as an example in the rest of paper) outbreak, there is only a very few number of individuals infected by SARS, and the medical resources are enough at this stage. Meanwhile, the early symptoms are very similar to flu and there is no effective way to identify the infected patients. Thus, patients infected with different viruses can be got treatment in different areas of the hospital at the same time.

With the growing numbers of SARS infected cases, those pose a grave threat to public health. Meanwhile, various kinds of control strategies have constraints based on the limited medical resources such as doctors, vaccines, drugs, hospital beds, isolation places, medical devices, and so on, especially in rural areas in many developing countries [6–10]. The medical resource limitation seriously restricts the prevention and treatment for SARS. At this moment, the department of health or state has to cite the urgency of fighting SARS, adopts green passage policy that speeds for isolation and treatment for SARS, so the doctors have to focus their attentions on the SARS infected cases only. For the patients with common flu, the doctors can only prescribe medicines and advise them to go home for home treatment, which can significantly relieve the pressure of limited medical resources on hospital or doctors.

In order to describe the effects of limited medical resource and selection strategy discussed above, the number of the patients infected by SARS in a compartment has been chosen as an index for hospital or doctors to use decisions. That is, if the number of the patients infected by SARS is below the threshold level which can be determined analytically (see main text for more details), there is no limited medical resource and selection pressure; above the threshold, due to the limited resource, and doctors treat SARS only. This type of control strategy is called as threshold control policy [11,12], which can be described by Filippov systems [13,14]. Recently, non-smooth Filippov infectious disease models have been investigated by many researchers [10,15–18].

In the present work, a non-smooth Filippov infectious disease model with threshold strategy induced by selective treatment measure is derived. The sliding mode dynamics and the existence of all types equilibria have been discussed. Numerical sliding bifurcation analyses show that the proposed Filippov system has rich sliding bifurcations. The key factors which affect the selective treatment measures and the threshold value of infected cases for emerging infectious disease have been discussed in more detail. Our main results show that reducing the threshold value to an appropriate level could contribute to the efficacy on prevention and treatment of emerging infectious disease, which indicates that the selection pressures can be beneficial to prevent the emerging infectious disease under medical resource limitation.

## 2. Models and threshold level

Let  $S(t)$ ,  $I_1(t)$ ,  $I_2(t)$  and  $R(t)$  denote the numbers of susceptible, the patients with SARS, common flu and recovered individuals at time  $t$ , respectively. For simplification, we assume that the people can only be infected either by SARS virus or by common flu virus. Further, based on the classical infectious disease model with limited capacity for treatment [9,10,19–25] we propose the following  $SI_1I_2R$  model as the basic model in this study

$$\begin{cases} \dot{S}(t) = A - \mu_S S - \beta_1 SI_1 - \beta_2 SI_2, \\ \dot{I}_1(t) = \beta_1 SI_1 - \mu_1 I_1 - \nu_1 I_1 - \frac{p_1 c_1 I_1}{1 + p_1 b_1 I_1 + p_2 b_2 I_2}, \\ \dot{I}_2(t) = \beta_2 SI_2 - \mu_2 I_2 - \nu_2 I_2 - \frac{p_2 c_2 I_2}{1 + p_1 b_1 I_1 + p_2 b_2 I_2}, \\ \dot{R}(t) = \nu_1 I_1 + \nu_2 I_2 + \frac{p_1 c_1 I_1 + p_2 c_2 I_2}{1 + p_1 b_1 I_1 + p_2 b_2 I_2} - \mu_R R. \end{cases} \quad (2.1)$$

The parameters of model (2.1) are summarized in Table 1. Obviously, two probabilities  $p_1, p_2 \in [0, 1]$ . Note that some very special cases of model (2.1) have been studied in our previous work [10], and the main purpose in this work is to investigate the generalized cases and reveal the rich dynamics and important biological implications concerning emerging infectious disease control.

It follows from model (2.1) that the total recovery rate

$$\frac{p_1 c_1 I_1 + p_2 c_2 I_2}{1 + p_1 b_1 I_1 + p_2 b_2 I_2} \quad (2.2)$$

is the major concern for the doctors once the emerging infectious disease outbreaks. Intuitively, how to choose the treatment proportions  $p_1$  and  $p_2$  for the patients infected by different virus such that the total recovery rate reaches its maximal value? To address this question, we discuss the selective strategies in the following.

Conditional upon resource limitation, we first assume that medical treatment service for SARS is more than the patients infected by flu, that is

$$\frac{c_1}{b_1} > \frac{c_2}{b_2}. \quad (2.3)$$

**Table 1**  
Parameter and definition for model (2.1).

| Parameters | Definitions for epidemic dynamics                                  |
|------------|--|
| $A$        | Recruitment rate of susceptible individuals                        |
| $\beta_1$  | The basic transmission coefficient of SARS                         |
| $\beta_2$  | Basic transmission coefficient of flu                              |
| $\mu_S$    | Natural death rates of susceptible individuals                     |
| $\mu_R$    | Natural death rates of recovered individuals                       |
| $\mu_1$    | Disease-related and natural death of the patients with SARS        |
| $\mu_2$    | Disease-related and natural death of the patients with flu         |
| $\nu_1$    | Natural recovery rate for SARS                                     |
| $\nu_2$    | Natural recovery rate for flu                                      |
| $c_1$      | The maximal recovery rate per unit time for the patients with SARS |
| $c_2$      | The maximal recovery rate per unit time for the patients with flu  |
| $b_1$      | Delayed effects on the treatment for the patients with SARS        |
| $b_2$      | Delayed effects on the treatment for the patients with flu         |
| $p_1$      | Probability that doctors treat the patients with SARS              |
| $p_2$      | Probability that doctors treat the patients with flu               |

Based on the total recovery rate defined by (2.2), we define the function  $\mathcal{F}$  with respect to  $p_1$  and  $p_2$  as follows

$$\mathcal{F}(p_1, p_2) = \frac{p_1 c_1 I_1 + p_2 c_2 I_2}{1 + p_1 b_1 I_1 + p_2 b_2 I_2}. \quad (2.4)$$

Taking the derivatives of the function  $\mathcal{F}$  with respect to  $p_1$  and  $p_2$  respectively, one yields

$$\frac{\partial \mathcal{F}}{\partial p_1} = \frac{[c_1 + p_2 I_2 (b_2 c_1 - b_1 c_2)] I_1}{(1 + p_1 b_1 I_1 + p_2 b_2 I_2)^2}, \quad \frac{\partial \mathcal{F}}{\partial p_2} = \frac{[c_2 - p_1 I_1 (b_2 c_1 - b_1 c_2)] I_2}{(1 + p_1 b_1 I_1 + p_2 b_2 I_2)^2},$$

and it follows from inequality (2.3) that  $\frac{\partial \mathcal{F}}{\partial p_1} > 0$ , and consequently the function  $\mathcal{F}$  is monotonically increasing with respect to  $p_1$ . According to the sign of the function  $\frac{\partial \mathcal{F}}{\partial p_2}$  with respect to  $I_1$ , we address the following interesting results concerning the selection strategies. To show this, we define the threshold value for SARS patients as follows

$$I_c \triangleq \frac{c_2}{b_2 c_1 - b_1 c_2} \quad (2.5)$$

which decides whether the hospital carries out the selective strategy or not. Thus, there are two cases:

□ If  $I_1 < I_c$ , the function  $\mathcal{F}$  can obtain the maximum value at

$$p_1 = 1, \quad p_2 = 1.$$

□ If  $I_1 > I_c$ , the function  $\mathcal{F}$  can obtain the maximum value at

$$p_1 = 1, \quad p_2 = 0.$$

Therefore, taking into account above facts, in the early stages of SARS outbreak ( $I_1 < I_c$ ), the patients with flu can be treated simultaneously with SARS, i.e.,  $p_1 = p_2 = 1$ . Then model (2.1) becomes

$$\begin{cases} \dot{S}(t) = A - \mu_S S - \beta_1 S I_1 - \beta_2 S I_2, \\ \dot{I}_1(t) = \beta_1 S I_1 - \mu_1 I_1 - \nu_1 I_1 - \frac{c_1 I_1}{1 + b_1 I_1 + b_2 I_2}, \\ \dot{I}_2(t) = \beta_2 S I_2 - \mu_2 I_2 - \nu_2 I_2 - \frac{c_2 I_2}{1 + b_1 I_1 + b_2 I_2}, \\ \dot{R}(t) = \nu_1 I_1 + \nu_2 I_2 + \frac{c_1 I_1 + c_2 I_2}{1 + b_1 I_1 + b_2 I_2} - \mu_R R. \end{cases} \quad (2.6)$$

With the increasing number of infected cases with SARS ( $I_1 > I_c$ ), the doctors have to isolate and treat SARS patients only, i.e.,  $p_1 = 1$ ,  $p_2 = 0$ . Thus model (2.1) becomes

$$\begin{cases} \dot{S}(t) = A - \mu_S S - \beta_1 S I_1 - \beta_2 S I_2, \\ \dot{I}_1(t) = \beta_1 S I_1 - \mu_1 I_1 - \nu_1 I_1 - \frac{c_1 I_1}{1 + b_1 I_1}, \\ \dot{I}_2(t) = \beta_2 S I_2 - \mu_2 I_2 - \nu_2 I_2, \\ \dot{R}(t) = \nu_1 I_1 + \nu_2 I_2 + \frac{c_1 I_1}{1 + b_1 I_1} - \mu_R R. \end{cases} \quad (2.7)$$

To simplify models (2.6) and (2.7), we assume that the number of the patients infected by flu each year is a constant, i.e.,  $I_2 = k \in \mathbb{Z}^+$ . Thus, models (2.6) and (2.7) can be rewritten as the following Filippov system [13,14]

$$\begin{cases} \dot{S}(t) = A - \mu S - \beta SI, \\ \dot{I}(t) = \beta SI - \nu I - \frac{c_1 I}{1 + b_1 I + \varepsilon b_2 k} \end{cases} \quad (2.8)$$

with

$$\varepsilon = \begin{cases} 1, & H(Z) < 0, \\ 0, & H(Z) > 0, \end{cases} \quad (2.9)$$

where  $\mu = \mu_S + \beta_2 k$ ,  $\nu = \mu_1 + \nu_1$ ,  $I_c = c_2/(b_2 c_1 - b_1 c_2)$ ,  $\beta = \beta_1$ ,  $I = I_1$  and  $H(Z) = I - I_c$  with vector  $Z = (S, I)^T$ . Model (2.8) with (2.9) is a description of the threshold control policy, which is referred to as an on-off control, see [11,12] for more detailed discussion on Filippov system. Note that the special case (i.e.  $b_1 = 0$ ) has been investigated recently [10].

Let

$$F_{G_1}(Z) = \left( A - \mu S - \beta SI, \beta SI - \nu I - \frac{c_1 I}{1 + b_1 I + b_2 k} \right)^T, \\ F_{G_2}(Z) = \left( A - \mu S - \beta SI, \beta SI - \nu I - \frac{c_1 I}{1 + b_1 I} \right)^T.$$

Then model (2.8) with (2.9) can be rewritten as the following generalized Filippov system [13,14]

$$\dot{Z}(t) = \begin{cases} F_{G_1}(Z), & Z \in G_1, \\ F_{G_2}(Z), & Z \in G_2, \end{cases} \quad (2.10)$$

where

$$G_1 = \{Z \in \mathbb{R}_+^2 | H(Z) < 0\}, \quad G_2 = \{Z \in \mathbb{R}_+^2 | H(Z) > 0\}.$$

Furthermore, the discontinuity boundary (or manifold)  $\Sigma$  separating two regions  $G_1$  and  $G_2$  is described as  $\Sigma = \{Z \in \mathbb{R}_+^2 | H(Z) = 0\}$  and  $H(Z)$  is a smooth scalar function with non-vanishing gradient  $H_Z(Z)$  on  $\Sigma$ .

The main characteristics of Filippov system (2.10) is that selective strategy is suppressed when the number of patients infected by SARS (i.e.,  $I(t)$ ) is below a previously chosen threshold policy  $I_c$ . With this number  $I(t)$  increases and exceeds the threshold  $I_c$ , the hospital will treat severe cases only since the shortage of medical resources, that is, the selective strategy is implemented.

The following definitions on all types of equilibria of Filippov system (2.10) [26,27] are necessary throughout the paper.

**Definition 2.1.** A point  $Z_*$  is called a real equilibrium of Filippov system (2.10) if  $F_{G_1}(Z_*) = 0, H(Z_*) < 0$ , or  $F_{G_2}(Z_*) = 0, H(Z_*) > 0$ . Similarly, a point  $Z_*$  is called a virtual equilibrium if  $F_{G_1}(Z_*) = 0, H(Z_*) > 0$ , or  $F_{G_2}(Z_*) = 0, H(Z_*) < 0$ . Both the real and virtual equilibria are called regular equilibria.

**Definition 2.2.** A point  $Z_*$  is called a pseudo-equilibrium if it is an equilibrium of the sliding mode of system (2.10), i.e.,  $\alpha(Z)F_{G_1}(Z_*) + (1 - \lambda)F_{G_2}(Z_*) = 0, H(Z_*) = 0$  and  $0 < \alpha(Z) < 1$ , and

$$\alpha(Z) = \frac{\langle H_Z(Z), F_{G_2}(Z) \rangle}{\langle H_Z(Z), F_{G_2}(Z) - F_{G_1}(Z) \rangle},$$

where  $\langle \cdot, \cdot \rangle$  denotes the standard scalar product.

**Definition 2.3.** A point  $Z_*$  is called a boundary equilibrium of Filippov system (2.10) if  $H(Z_*) = 0$  and  $F_{G_1}(Z_*) = 0$  (or  $F_{G_2}(Z_*) = 0$ ).

**Definition 2.4.** A point  $Z_*$  is called a tangency point of Filippov system (2.10) if  $Z_* \in \Sigma_s$  and  $F_{G_1}H(Z_*) = 0$  (or  $F_{G_2}H(Z_*) = 0$ ), where  $F_{G_i}H(Z) = \langle H_Z(Z), F_{G_i}(Z) \rangle$  is the Lie derivative [28] for  $i = 1, 2$ .

### 3. Qualitative analysis of two subsystems

If  $I < I_c$ , then the following system plays a key role in analyzing the Filippov system (2.10)

$$\begin{cases} \dot{S}(t) = A - \mu S - \beta SI, \\ \dot{I}(t) = \beta SI - \nu I - \frac{c_1 I}{1 + b_2 k + b_1 I}, \end{cases} \quad (3.1)$$

and the basic reproduction number of subsystem (3.1) reads

$$\mathcal{R}_1 = \frac{\beta}{\nu + \frac{c_1}{1 + b_2 k}} \frac{A}{\mu}.$$

It is obvious that subsystem (3.1) always has a unique disease-free equilibrium  $E_1^0 = (A/\mu, 0)$ , which is globally asymptotically stable if  $\mathcal{R}_1 < 1$ .

The endemic equilibria of subsystem (3.1) are solutions of

$$\begin{cases} A - \mu S - \beta SI = 0, \\ \beta S - \nu - \frac{c_1}{1+b_2k+b_1I} = 0, \end{cases}$$

which yields

$$I^2 + m_1 I + n_1 = 0, \quad (3.2)$$

where

$$m_1 = \frac{\nu(\beta(1+b_2k) + \mu b_1) + c_1\beta - A\beta b_1}{\nu\beta b_1}, \quad n_1 = \frac{\nu\mu(1+b_2k) + c_1\mu}{\nu\beta b_1}(1 - \mathcal{R}_1).$$

Noting that  $n_1 < 0$  if and only if  $\mathcal{R}_1 > 1$ ,  $n_1 = 0$  if and only if  $\mathcal{R}_1 = 1$ ,  $n_1 > 0$  if and only if  $\mathcal{R}_1 < 1$ .

For the simplicity and convenience of exposition, we denote

$$I_1^1 = \frac{-m_1 + \sqrt{\Delta_1}}{2}, \quad I_1^2 = \frac{-m_1 - \sqrt{\Delta_1}}{2}, \quad \Delta_1 = m_1^2 - 4n_1, \quad S_1^i = \frac{A}{\mu + \beta I_1^i}, \quad i = 1, 2.$$

**Lemma 3.1.** For subsystem (3.1), we have

- (1) If  $\mathcal{R}_1 > 1$ , there exists a unique endemic equilibrium  $E_1^1 = (S_1^1, I_1^1)$ ;
- (2) If  $\mathcal{R}_1 = 1$  and  $m_1 < 0$ , there exists a unique endemic equilibrium  $E_1 = (S_1^1, I_1^1)$ ;
- (3) If  $\mathcal{R}_1 = 1$  and  $m_1 \geq 0$ , there exists no endemic equilibrium;
- (4) If  $\mathcal{R}_1 < 1$ ,  $m_1 < 0$  and  $\Delta_1 > 0$ , there exists two endemic equilibria  $E_1^1 = (S_1^1, I_1^1)$  and  $E_1^2 = (S_1^2, I_1^2)$ ;
- (5) If  $\mathcal{R}_1 < 1$ ,  $m_1 < 0$  and  $\Delta_1 = 0$ , two endemic equilibria  $E_1^1 = (S_1^1, I_1^1)$  and  $E_1^2 = (S_1^2, I_1^2)$  coalesce at a unique endemic equilibrium of multiplicity 2;
- (6) If  $\mathcal{R}_1 < 1$ ,  $m_1 < 0$  and  $\Delta_1 < 0$ , there exists no endemic equilibrium;
- (7) If  $\mathcal{R}_1 < 1$  and  $m_1 \geq 0$ , there exists no endemic equilibrium.

Next, we will study the stability of the endemic equilibria of subsystem (3.1). The characteristic equation about the endemic equilibrium  $E_1^i(S_1^i, I_1^i)$ ,  $i = 1, 2$  is given by

$$\lambda^2 + H(I_1^i)\lambda + G(I_1^i) = 0, \quad (3.3)$$

where

$$H(I_1^i) = \mu + \beta I_1^i - \frac{c_1 b_1 I_1^i}{(1 + b_1 I_1^i + b_2 k)^2}, \quad G(I_1^i) = \frac{A\beta^2 I_1^i}{\mu + \beta I_1^i} - \frac{(\mu + \beta I_1^i)c_1 b_1 I_1^i}{(1 + b_1 I_1^i + b_2 k)^2}. \quad (3.4)$$

We can obtain the following lemmas from (3.3).

**Lemma 3.2.** If  $\mathcal{R}_1 > 1$ , then the endemic equilibrium  $E_1^1$  of system (3.1) is a stable node or focus when  $H(I_1^1) > 0$ ;  $E_1^1$  is an unstable node or focus when  $H(I_1^1) < 0$ , and system (3.1) has at least one closed orbit in  $\Omega$ ;  $E_1^1$  is a center of the linear system when  $H(I_1^1) = 0$ .

**Lemma 3.3.** If  $\mathcal{R}_1 < 1$ ,  $m_1 < 0$  and  $\Delta_1 > 0$  and  $A > A_1$ , then the endemic equilibrium  $E_1^2$  of system (3.1) is a saddle; and  $E_1^1$  is an unstable node or focus when  $H(I_1^1) < 0$ ;  $E_1^1$  is a center of linear system when  $H(I_1^1) = 0$ .

The methods of proving Lemmas 3.1–3.3 are similar to those in Refs. [23,24], see these references for more details.

If  $I > I_c$ , then the Filippov system (2.10) becomes

$$\begin{cases} \dot{S}(t) = A - \mu S - \beta SI, \\ \dot{I}(t) = \beta SI - \nu I - \frac{c_1 I}{1+b_1 I}, \end{cases} \quad (3.5)$$

and it has a unique disease-free equilibrium  $E_2^0 = (A/\mu, 0)$ , which is globally asymptotically stable if  $\mathcal{R}_2 < 1$ . Here,

$$\mathcal{R}_2 = \frac{\beta}{\nu + c_1} \frac{A}{\mu}$$

is the basic reproduction number of subsystem (3.5).

For convenience, we denote

$$E_2^1 = (S_2^1, I_2^1) = \left( \frac{A}{\mu + \beta I_2^1}, \frac{-m_2 + \sqrt{\Delta_2}}{2} \right), \quad E_2^2 = (S_2^2, I_2^2) = \left( \frac{A}{\mu + \beta I_2^2}, \frac{-m_2 - \sqrt{\Delta_2}}{2} \right),$$

where

$$m_2 = \frac{\nu(\beta + \mu b_1) + c_1 \beta - A \beta b_1}{\nu \beta b_1}, \quad n_2 = \frac{\nu \mu + c_1 \mu}{\nu \beta b_1} (1 - \mathcal{R}_2), \quad \Delta_2 = m_2^2 - 4n_2.$$

The characteristic equation about  $E_2^i (S_2^i, I_2^i)$  for  $i = 1, 2$  is given by

$$\lambda^2 + H(I_2^i) \lambda + G(I_2^i) = 0,$$

where

$$H(I_2^i) = \mu + \beta I_2^i - \frac{c_1 b_1 I_2^i}{(1 + b_1 I_2^i)^2}, \quad G(I_2^i) = \frac{A \beta^2 I_2^i}{\mu + \beta I_2^i} - \frac{(\mu + \beta I_2^i) c_1 b_1 I_2^i}{(1 + b_1 I_2^i)^2}.$$

Similar conclusions as Lemmas 3.1–3.3 for subsystem (3.5) can be obtained, and those are not described here.

#### 4. Basic properties of Filippov system (2.10)

##### 4.1. Existence of sliding domain

It follows from Filippov convex method [13,14] that one can define the sliding vector field as a convex combination of the two vector fields

$$F_G(Z) = \alpha(Z) F_{G_1}(Z) + (1 - \alpha(Z)) F_{G_2}(Z) \quad (4.1)$$

with

$$\alpha(Z) = \frac{\langle H_Z(Z), F_{G_2}(Z) \rangle}{\langle H_Z(Z), F_{G_2}(Z) - F_{G_1}(Z) \rangle}, \quad 0 \leq \alpha(Z) \leq 1.$$

Noting that the control  $\alpha(Z) = 1$  indicates that the flow is governed by  $F_{G_1}$  alone, which must be tangent to the switching surface  $\Sigma$ . Analogously,  $\alpha(Z) = 0$  shows a tangency of flow  $F_{G_2}$  with  $\Sigma$ . Therefore, the sliding subset can be defined as

$$\Sigma_s = \{Z \in \Sigma \mid 0 \leq \alpha(Z) \leq 1\}. \quad (4.2)$$

The boundaries of the sliding subset are  $\partial \Sigma_s^+ = \{Z \in \Sigma : \alpha(Z) = 0\}$  and  $\partial \Sigma_s^- = \{Z \in \Sigma : \alpha(Z) = 1\}$  where one of the vector fields is tangent to  $\Sigma$  at the boundaries  $\partial \Sigma_s^+$  or  $\partial \Sigma_s^-$ . In particular, we can use the signs of  $\langle H_Z(Z), F_{G_i}(Z) \rangle$ ,  $i = 1, 2$  to define the sewing region, escaping region and sliding region, see Refs. [29,30] for more details.

By simple calculation, we have

$$\alpha(Z) = \frac{(1 + b_1 I_c + b_2 k) [c_1 - (\beta S - \nu)(1 + b_1 I_c)]}{c_1 b_2 k},$$

solving the inequality  $0 \leq \alpha(Z) \leq 1$  with respect to  $S$ , one yields

$$\frac{1}{\beta} \left( \frac{c_1}{1 + b_1 I_c + b_2 k} + \nu \right) \leq S \leq \frac{1}{\beta} \left( \frac{c_1}{1 + b_1 I_c} + \nu \right), \quad I_c = \frac{c_2}{b_2 c_1 - b_1 c_2}.$$

Therefore, the sliding segment of Filippov (2.10) can be defined as

$$\Sigma_s = \left\{ (S, I_c) \mid \frac{1}{\beta} \left( \frac{c_1}{1 + b_1 I_c + b_2 k} + \nu \right) \leq S \leq \frac{1}{\beta} \left( \frac{c_1}{1 + b_1 I_c} + \nu \right), \quad I_c = \frac{c_2}{b_2 c_1 - b_1 c_2} \right\}. \quad (4.3)$$

##### 4.2. Sliding mode dynamics

Here we employ Utkin's equivalent control method introduced in [14] to obtain the differential equation for sliding dynamics defined in the region  $\Sigma_s$ . It follows from  $H = 0$  that

$$\frac{\partial H}{\partial Z} = I'_1 = \beta S I_c - \nu I_c - \frac{c_1 I_c}{1 + b_1 I_c + \varepsilon b_2 k} = 0,$$

and solving the above equation with respect to  $\varepsilon$  yields

$$\varepsilon = \frac{c_1 - (1 + b_1 I_c)(\beta S - \nu)}{b_2 k(\beta S - \nu)}.$$

According to Utkin's equivalent control method, we can obtain the dynamics defined in  $\Sigma_s$  which can be determined by the following scalar differential equation

$$\dot{S}(t) = A - \mu S - \beta S I_c, \quad I_c = \frac{c_2}{b_2 c_1 - b_1 c_2}. \quad (4.4)$$

#### 4.3. Equilibria of Filippov system (2.10)

For the simplicity and convenience of exposition, we denote real equilibrium as  $E_R$ , virtual equilibrium as  $E_V$ , pseudo-equilibrium as  $E_P$ , boundary equilibrium as  $E_B$  and tangent point as  $E_T$ , respectively.

**Regular equilibrium:** For convenience, we just consider the subsystem has two endemic equilibria. From Lemma 3.1, subsystem (3.1) has two endemic equilibria  $E_1^1 = (S_1^1, I_1^1)$  and  $E_1^2 = (S_1^2, I_1^2)$  provided that  $\mathcal{R}_1 < 1$ ,  $m_1 < 0$  and  $\Delta_1 > 0$ . If  $I_1^1 < I_c$ , then both  $E_1^1$  and  $E_1^2$  are real equilibria for subsystem (3.1), denoted by  $E_R^{11}$  and  $E_R^{12}$ ; While if  $I_1^1 > I_c$ , then both  $E_1^1$  and  $E_1^2$  become virtual equilibria for subsystem (3.1), denoted by  $E_V^{11}$  and  $E_V^{12}$ ; Else if  $I_1^2 < I_c < I_1^1$ , then the equilibria  $E_1^1$  and  $E_1^2$  become the virtual and real equilibria, denoted by  $E_V^{11}$  and  $E_R^{12}$ , respectively.

Analogously, subsystem (3.5) has two endemic equilibria  $E_2^i = (S_2^i, I_2^i)$  for  $i = 1, 2$  as  $\mathcal{R}_2 < 1$ ,  $m_2 < 0$  and  $\Delta_2 > 0$ . And  $E_2^i$  could be a real or virtual equilibrium (denoted by  $E_R^{2i}$  or  $E_V^{2i}$ , respectively) which depends on the size of  $I_2^i$  and  $I_c$ .

**Pseudo-equilibrium:** For the existence of pseudo-equilibrium, we denote  $E_P = (S_P, I_c)$ , according to (4.4) the  $S_P$  component of the pseudo-equilibrium of sliding flow satisfies the following equation

$$A - \mu S_P - \beta S_P I_c = 0,$$

where  $S_P = A/(\mu + \beta I_c) \in \Sigma_s$ , and  $S_P$  is a unique positive steady state of (4.4). That is, if  $S_P = A/(\mu + \beta I_c) \in \Sigma_s$  holds true, Filippov system (2.10) exists a unique pseudo-equilibrium  $E_P$ .

For the stability of pseudo-equilibrium  $E_P = (S_P, I_c)$ , we rewrite sliding mode equation (4.4) as

$$\dot{S}(t) = A - \mu S - \beta S I_c \doteq \phi(S), \quad (4.5)$$

it follows from (4.5) that

$$\left. \frac{\partial \phi(S)}{\partial S} \right|_{S=S_P} = -\mu - \beta I_c < 0,$$

which indicates that the pseudo-equilibrium  $E_P$  is locally stable in  $\Sigma_s$ .

**Boundary equilibrium:** The boundary equilibria of Filippov system (2.10) satisfy equations

$$\begin{cases} A - \mu S - \beta S I_c = 0, \\ \beta S - v - \frac{c_1}{1+b_1 I_c + \varepsilon b_2 k} = 0, \\ I_c = \frac{c_2}{c_1 b_2 - c_2 b_1}, \end{cases}$$

which indicates that if

$$\frac{A}{\mu + \beta I_c} = \frac{1}{\beta} \left( \frac{c_1}{1 + b_1 I_c + \varepsilon b_2 k} + v \right), \quad (4.6)$$

then we have boundary equilibrium  $E_B = (\frac{1}{\beta} [\frac{c_1}{1+b_1 I_c + \varepsilon b_2 k} + v], I_c)$ . We denote  $E_B^{11}$  or  $E_B^{12}$  provided (4.6) as  $\varepsilon = 1$ , i.e.,  $I_c = I_1^1$  or  $I_c = I_1^2$  (if  $I_1^1$  or  $I_1^2$  exists), and  $E_B^{21}$  or  $E_B^{22}$  provided (4.6) as  $\varepsilon = 0$ , i.e.,  $I_c = I_2^1$  or  $I_c = I_2^2$  (if  $I_2^1$  or  $I_2^2$  exists).

**Tangent point:** According to Definition 2.4, the tangent point  $E_T$  on sliding segment  $\Sigma_s$  satisfies equations

$$\begin{cases} \beta S - v - \frac{c_1}{1+b_1 I_c + b_2 k} = 0, \\ I_c = \frac{c_2}{c_1 b_2 - c_2 b_1} \end{cases}$$

and

$$\begin{cases} \beta S - v - \frac{c_1}{1+b_1 I_c} = 0, \\ I_c = \frac{c_2}{c_1 b_2 - c_2 b_1}. \end{cases}$$

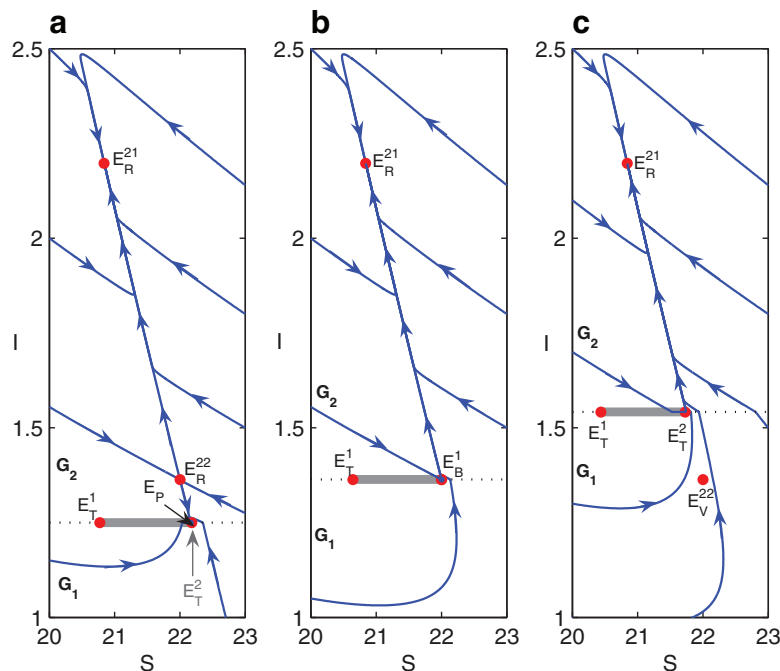
Hence, there may be two possible tangent points including  $E_T^1 = (\frac{1}{\beta} [\frac{c_1}{1+b_1 I_c + b_2 k} + v], I_c)$  and  $E_T^2 = (\frac{1}{\beta} [\frac{c_1}{1+b_1 I_c} + v], I_c)$ .

The richness of equilibria of Filippov system (2.10) could result in a number of equilibrium bifurcations as the key parameter varies. Thus, in the coming section, we would like to investigate the local and global sliding bifurcations of system (2.10).

## 5. Sliding bifurcation analysis of Filippov system (2.10)

By employing numerical methods to investigate the qualitative behaviors of Filippov system (2.10), the sliding bifurcation analyses including local and global bifurcations of one parameter Filippov system (2.10) are provided in this section. Therefore, we fix all other parameters and choose the maximal recovery rate for the patients with flu (i.e.,  $c_2$ ) as a bifurcation parameter, noting that  $c_2$  is directly related to the threshold value  $I_c$ .





**Fig. 1.** Boundary-saddle bifurcation for Filippov system (2.10). Parameters are:  $A = 363$ ,  $\beta = 1.1$ ,  $\mu = 15$ ,  $\nu = 13.2$ ,  $b_1 = 0.2$ ,  $c_1 = 14$ ,  $k = 1$ ,  $b_2 = 0.2$ , and (A)  $c_2 = 2.8$ ; (B)  $c_2 = 3$ ; (C)  $c_2 = 3.3$ .

### 5.1. Local sliding bifurcation

Throughout this section, we will investigate the local sliding bifurcation of Filippov system (2.10). Boundary equilibrium bifurcation, as a type of local sliding bifurcation in Filippov system, is characterized by the collision of pseudo-equilibrium, tangent point, and real equilibrium (or tangent point and real equilibrium) at the discontinuity surface when one parameter passes through a critical value, and includes boundary node, focus and saddle bifurcations which will be addressed in the following.

Noting that the boundary equilibrium bifurcation occurs at  $E_B$  if  $F_{G_i}(E_B)$  is invertible (or equivalently the eigenvalues of  $DF_{G_i}(E_B)$  have real part different from zero and  $\langle H_Z(E_B), F_{G_j}(E_B) \rangle \neq 0$ ,  $i, j = 1, 2$ ;  $i \neq j$ ). Here the symbol  $DF_{G_i}(E_B)$  is the characteristic polynomial of subsystems (3.1) or (3.5) about the boundary equilibrium  $E_B$ . These bifurcations are classified as boundary saddle, boundary node and boundary focus in [31]. It follows from Section 4.2 that Filippov system (2.10) maybe have boundary equilibria  $E_B^{1i}$  and  $E_B^{2i}$  if  $I_1^i$  and  $I_2^i$  exist for  $i = 1, 2$ . For the boundary equilibria  $E_B^{11}$  and  $E_B^{12}$ , by simple calculations we have

$$F_{G_2}H(E_B^{11}) = \left( \beta S_1^1 - \nu - \frac{c_1}{1 + b_1 I_1^1} \right) I_1^1 < 0, \quad (5.1)$$

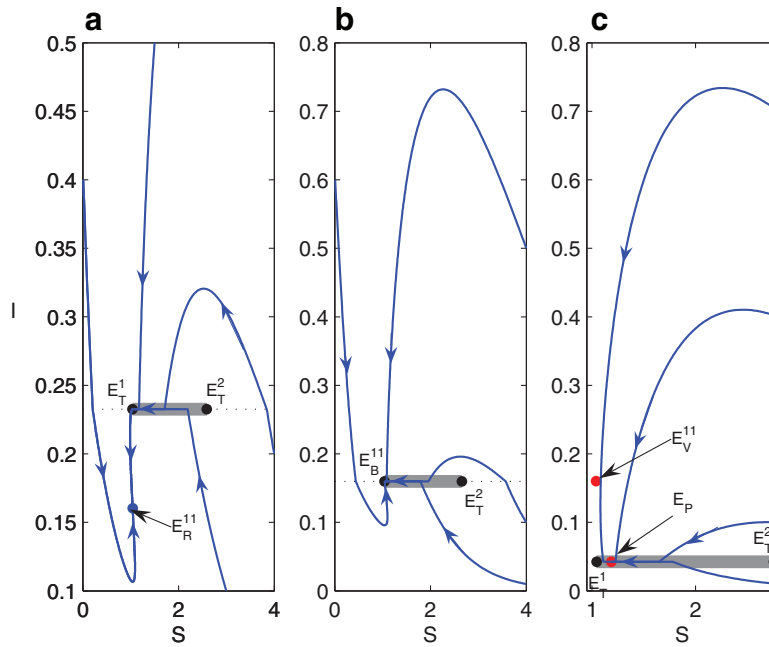
$$F_{G_2}H(E_B^{12}) = \left( \beta S_1^2 - \nu - \frac{c_1}{1 + b_1 I_1^2} \right) I_1^2 < 0, \quad (5.2)$$

and from Lemma 3.3, we know that  $DF_{G_i}(E_B)$  possesses complex eigenvalues with nonzero real part  $-H(I_1^i)/2$ ,  $i = 1, 2$  if  $E_B^{1i}$  is an saddle (a node, or a focus). Again, similar argument as above will yield for the boundary equilibria  $E_B^{21}$  and  $E_B^{22}$ . Hence, a boundary-equilibrium bifurcation occurs at  $E_B^{ij}$ ,  $i, j = 1, 2$ .

**Boundary-saddle bifurcation:** This type of bifurcation may occur for Filippov system (2.10) if three types of equilibria  $E_R$ ,  $E_T$  and  $E_P$  collide together simultaneously as parameter  $c_2$  passes through a critical value [31]. For example, when the parameter  $c_2$  passes through a critical value  $c_2^t \approx 3$ , a saddle  $E_R^{22}$ , a tangent point  $E_T^2$  and a pseudo-equilibrium  $E_P$  collide together, so the boundary-saddle bifurcation occurs at  $E_B^2$ , as shown in Fig. 1(B). In this case, the critical value

$$c_2^t = \frac{I_2^2 c_1 b_2}{1 + b_1 I_2^2}. \quad (5.3)$$

A saddle  $E_R^{22}$  (an unstable real equilibrium), a stable pseudo-equilibrium  $E_P$  and an invisible tangent  $E_T^2$  can coexist for  $c_2 < c_2^t$ , as shown in Fig. 1(A) with  $c_2 = 2.8$ . They collide together simultaneously as  $c_2 = c_2^t$  and are substituted by a virtual



**Fig. 2.** Boundary-node bifurcation for Filippov system (2.10). Parameters are:  $A = 1$ ,  $\beta = 1$ ,  $\mu = 0.8$ ,  $\nu = 0.8$ ,  $b_1 = 0.5$ ,  $c_1 = 2$ ,  $k = 6$ ,  $b_2 = 1.2$ , and (A)  $c_2 = 0.5$ ; (B)  $c_2 = 0.355$ ; (C)  $c_2 = 0.1$ .

equilibrium  $E_V^{22}$ , an invisible tangent  $E_T^2$ , while the pseudo-equilibrium  $E_P$  is disappeared, when the parameter  $c_2 > c_2^t$ , see Fig. 1(C) with  $c_2 = 3.3$  for more details.

**Boundary-node bifurcation:** From Fig. 2, we can see that the stable node  $E_R^{11}$  and a tangent point  $E_T^1$  collide together as the parameter  $c_2$  passes through the critical value  $c_2^t \approx 0.355$ , the boundary node bifurcation occurs at  $E_B^{11}$ , where the critical value  $c_2^t$  is

$$c_2^t = \frac{I_1^1 c_1 b_2}{1 + b_1 I_1^1}. \quad (5.4)$$

A stable node  $E_R^{11}$  and a tangent point  $E_T^1$  coexist, as shown in Fig. 2(A) with  $c_2 = 0.5$  when  $c_2 > c_2^t$ . They collide at  $c_2 = c_2^t$  (see Fig. 2(B)) and are substituted by a pseudo-equilibrium  $E_P$ , a tangent point  $E_T^1$  and a virtual equilibrium  $E_V^{11}$  as  $c_2 < c_2^t$ , see Fig. 2(C) with  $c_2 = 0.1$  for more details.

**Boundary-focus bifurcation:** Similarly, a boundary focus bifurcation of Filippov system (2.10) occurs at  $E_B^{21}$  as  $c_2^t \approx 0.635$ , see Fig. 3, and

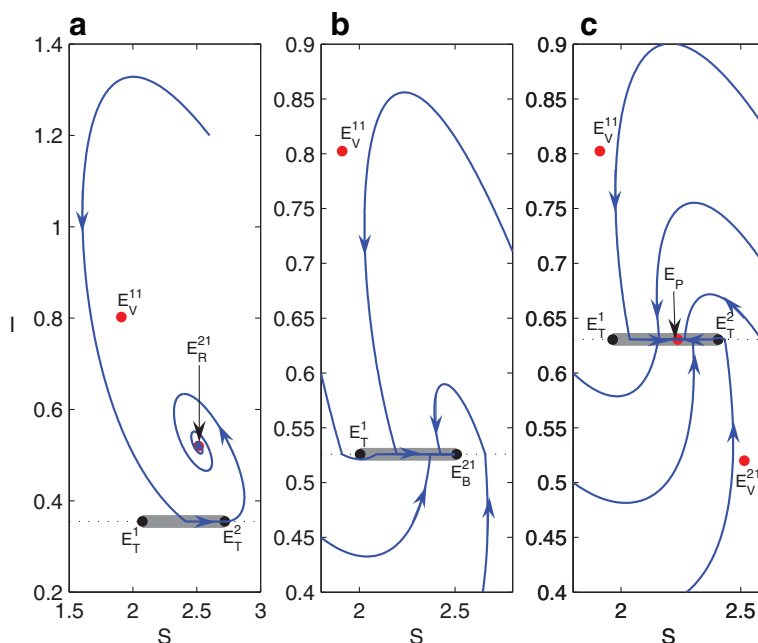
$$c_2^t = \frac{I_2^1 c_1 b_2}{1 + b_1 I_2^1}. \quad (5.5)$$

A stable focus  $E_R^{21}$  and a tangent point  $E_T^2$  coexist, as shown in Fig. 3(A) when  $c_2 < 0.635$ . They collide at  $c_2 = c_2^t$  (see Fig. 3(B) with  $c_2 = 0.5$ ) and are substituted by a pseudo-equilibrium  $E_P$ , a tangent point  $E_T^2$  and a virtual equilibrium  $E_V^{21}$  when  $c_2 > 0.635$ , as shown in Fig. 3(C) with  $c_2 = 0.7$ .

## 5.2. Global sliding bifurcation

Filippov system (2.10) could have standard periodic solutions that lie entirely in regions  $G_1$  or  $G_2$  through a Hopf bifurcation, respectively [31,32]. Meanwhile, as mentioned in Refs. [26,31], Filippov system (2.10) may have additional two types of new periodic solutions: periodic solutions which have a sliding segment in sliding segment  $\Sigma_s$  (i.e., sliding periodic solutions) and those which have only isolated points in common with  $\Sigma_s$  (i.e., crossing periodic solutions). Noting that a crossing periodic solution can pass through the boundary of the sliding segment  $\Sigma_s$ . Accordingly, the orbits corresponding to periodic solutions will be called standard, sliding and crossing cycles. In this section, we focus on the global sliding bifurcations such as grazing bifurcation (i.e., touching bifurcation), buckling bifurcation and crossing bifurcation.

**Grazing (or touching) bifurcation:** It follows from the Refs. [26,31], a standard periodic solution can collide with the sliding segments, and this type of bifurcation is called grazing or touching bifurcation. Noting that Filippov system (2.10) has a stable periodic solution in the interior of region  $G_2$ , as shown in Fig. 4(A) with  $c_2 = 0.5$ . At this moment, Filippov system (2.10) has two tangent points  $E_T^1$  and  $E_T^2$  lying on the boundary of the sliding mode, subsystem (3.5) has an unstable



**Fig. 3.** Boundary-focus bifurcation for Filippov system (2.10). Parameters are:  $A = 1.8$ ,  $\beta = 0.8$ ,  $\mu = 0.3$ ,  $\nu = 1$ ,  $b_1 = 1.5$ ,  $c_1 = 1.8$ ,  $k = 1$ ,  $b_2 = 1.2$ , and (A)  $c_2 = 0.5$ ; (B)  $c_2 = 0.635$ ; (C)  $c_2 = 0.7$ .

real equilibrium  $E_R^2$  while subsystem (3.1) has one unstable virtual equilibrium  $E_V^1$ . As the parameter  $c_2$  increases and passes through around 0.85, a grazing or touching bifurcation occurs, as shown in Fig. 4(B), which indicates that the standard period solution of Filippov system (2.10) collides with its tangent point  $E_T^2$ . As  $c_2$  continues to increase the cycle becomes a sliding cycle, where a piece of sliding segment belongs to the cycle, as shown in Fig. 4(C) with  $c_2 = 0.95$ .

Especially, as the bifurcation parameter  $c_2$  is increased to 1.6, the stable periodic cycle is disappeared, and pseudo-equilibrium  $E_P$  appears at  $c_2 = 1.6$ . Meanwhile, for subsystem (3.5), the real equilibrium  $E_R^2$  becomes a virtual equilibrium  $E_V^2$ , the real/virtual equilibrium bifurcation occurs at  $c_2 = 1.6$ , as shown in Fig. 4(D). Meanwhile, Fig. 4(D) also shows that pseudo-equilibrium of Filippov system cannot coexist with the real equilibria.

**Buckling bifurcation:** This type of bifurcation is defined as a standard piece of the cycle starts to pass the invisible quadratic tangent point as the bifurcation parameter varies [31], as shown in Fig. 5(C). Fig. 5(B) clearly shows that Filippov system (2.10) has a stable sliding periodic solution, an invisible quadratic tangent point  $E_T^1$ , an unstable regular equilibria  $E_V^1$  and  $E_R^2$ . As  $c_2$  increases and exceeds 0.7 the piece of the cycle starts to pass the invisible quadratic tangent point  $E_T^1$ , and consequently the cycle passes through the whole piece of the sliding segment  $\Sigma_s$ , and the sliding cycle does exist in region  $G_2$  and sliding segment  $\Sigma_s$ , see Fig. 5(C) for details. It follows from Fig. 5(B)–(D) that a buckling bifurcation occurs at  $c_2 = 1.05$ . Similarly, Fig. 5(F)–(H) also shows there exists a buckling bifurcation as  $c_2 = 2.7$ , unlike 5(B)–(D), the sliding cycle as shown in Fig. 5(G) does exist in region  $G_1$  and  $\Sigma_s$ .

**Crossing bifurcation:** Along with the variation of bifurcation parameter, a stable sliding periodic solution becomes a stable crossing periodic solution, this type bifurcation is called crossing bifurcation. It is interesting to note that a sliding cycle with a single sliding segment ending at  $E_T^1$  does exist in both regions  $G_1$ ,  $G_2$  and sliding segment  $\Sigma_s$ , see Fig. 5(H). As  $c_2$  increases and reaches 3.45 the sliding cycle only passes the tangent point  $E_T^1$  on the sliding segment  $\Sigma_s$ . At this moment, the stable sliding periodic solution becomes a stable crossing periodic solution, as shown in Fig. 5(I) with  $c_2 = 3.45$ . Fig. 5(J) shows that the sliding crossing cycle becomes a crossing cycle as  $c_2$  continues to increase and reach 4. Noting that the sliding segment  $\Sigma_s$  is located within the crossing cycle which exists in both regions  $G_1$  and  $G_2$ . It follows from Fig. 5(H)–(J) that a crossing bifurcation occurs at  $c_2 = 3.45$ . Similarly, Filippov system (2.10) must exist one crossing bifurcation from Fig. 5(A) to (B).

In summary, as bifurcation parameter  $c_2$  increases from 0.5 to 4 and all other parameters are fixed as those in Fig. 5, Filippov system (2.10) has rich sliding bifurcations, the following local and global sliding bifurcations occur sequentially: crossing  $\rightarrow$  buckling  $\rightarrow$  real/virtual equilibrium  $\rightarrow$  buckling  $\rightarrow$  crossing. Especially, as the bifurcation parameter  $c_2$  changes around 1.6, the stable periodic cycle is disappeared, and pseudo-equilibrium  $E_P$  appears at  $c_2 = 1.6$ , all the orbits tend to the pseudo-equilibrium  $E_P$ , which is locally asymptotically stable. There exists a real/virtual equilibrium bifurcation for Filippov system (2.10) in such case.

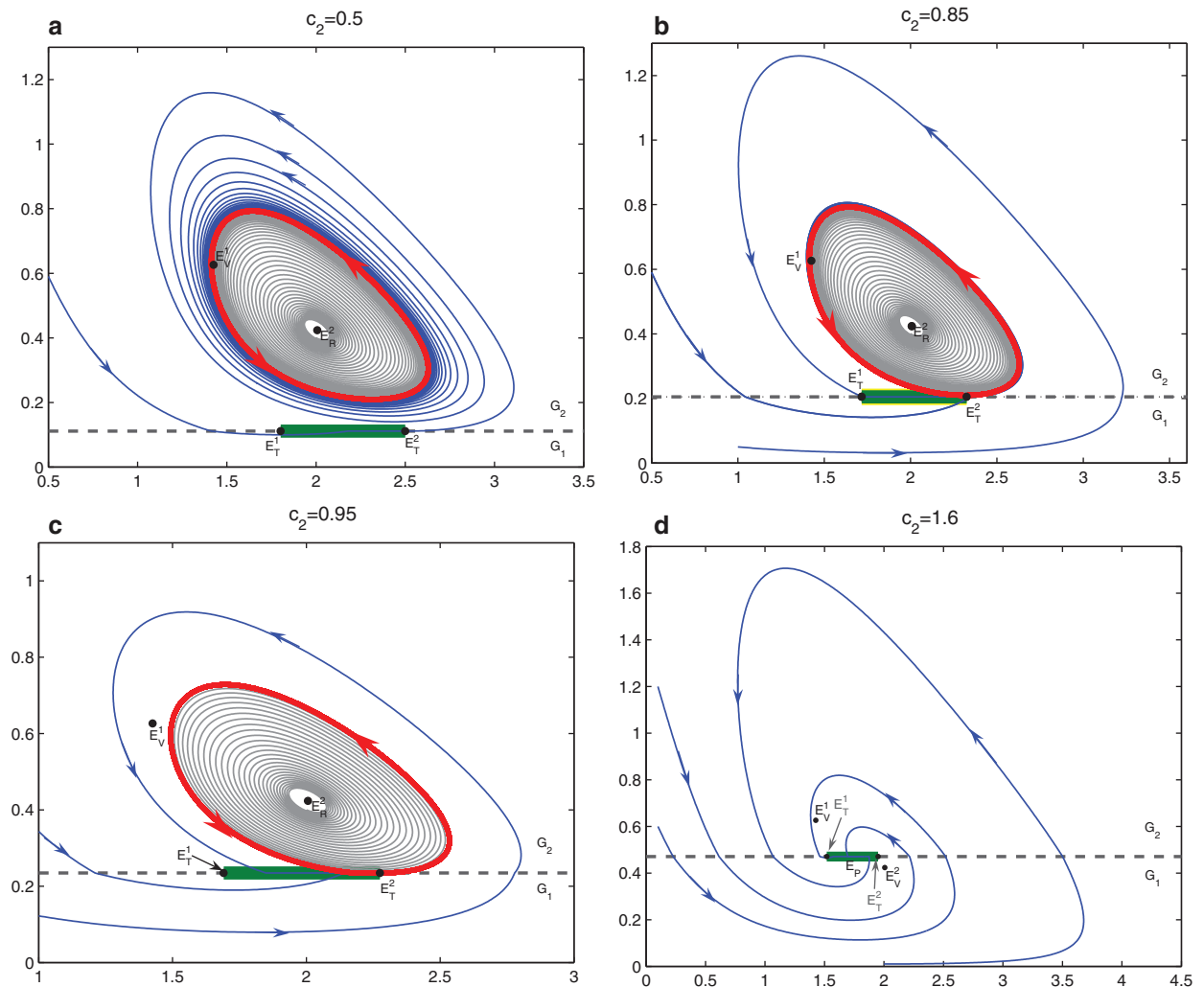


Fig. 4. Grazing (or touching) bifurcation for Filippov system (2.10). Parameters are:  $A = 2$ ,  $\nu = 0.5$ ,  $\beta = 2$ ,  $b_1 = 1$ ,  $b_2 = 1$ ,  $c_1 = 5$ ,  $k = 0.5$ ,  $\mu = 0.15$ .

## 6. Key parameters and biological significance

Previous analysis indicates that limited medical resources, the basic reproduction numbers  $\mathcal{R}_i$ ,  $i = 1, 2$  and threshold values  $I_c$  (or the selective strategy) are significant factors affecting the spread of the emerging infectious disease. In this section, we first investigate how the limited medical resources (i.e.,  $b_1$ ) affects the dynamic behaviors of Filippov system (2.10) or the spread of SARS.

As the medical resource limitation (i.e.,  $b_1 \neq 0$ ) is taken into account, then the dynamical behaviors of both subsystems (3.1) and (3.5) of Filippov system (2.10) become much more complex. Meanwhile, the dynamic behavior of the Filippov system (2.10) could be dramatically affected by the existence of medical resource limitation. In fact, the sliding mode could change as threshold value  $I_c$  changes, as shown in Fig. 6(A), the length of sliding segment is increased with growing recovery rate  $c_1$  from 0.6 to 1.5. Meanwhile, noting that Filippov system (2.10) does not have pseudo-equilibrium in this case  $b_1 = 0$ . However, with the medical resource limitation (i.e.,  $b_1 \neq 0$ ), there exists a pseudo-equilibrium  $E_p$ , which is globally stable with respect to the sliding segment  $\Sigma_s$ , as shown in Fig. 6(B). Those indicates that the emerging infectious disease will become endemic instead of elimination.

Meanwhile, as seen in Fig. 7, the rate of growth in the number of the patients with SARS is far more quickly before than after controlling. By comparing the red line with the blue one in Fig. 7, the limited medical resource has been a great influence on the time when the peak of patients appears and how long does the spread of SARS last. In fact, the blue line (with sufficient medical resources, i.e.,  $b_1 = 0$ ) shows that there is a peak of patients in rapid and a sharp decline, the spread of SARS lasts only for a short time. The red line (with limited resources, i.e.,  $b_1 = 2.5$ ) shows there is a delay in appearance

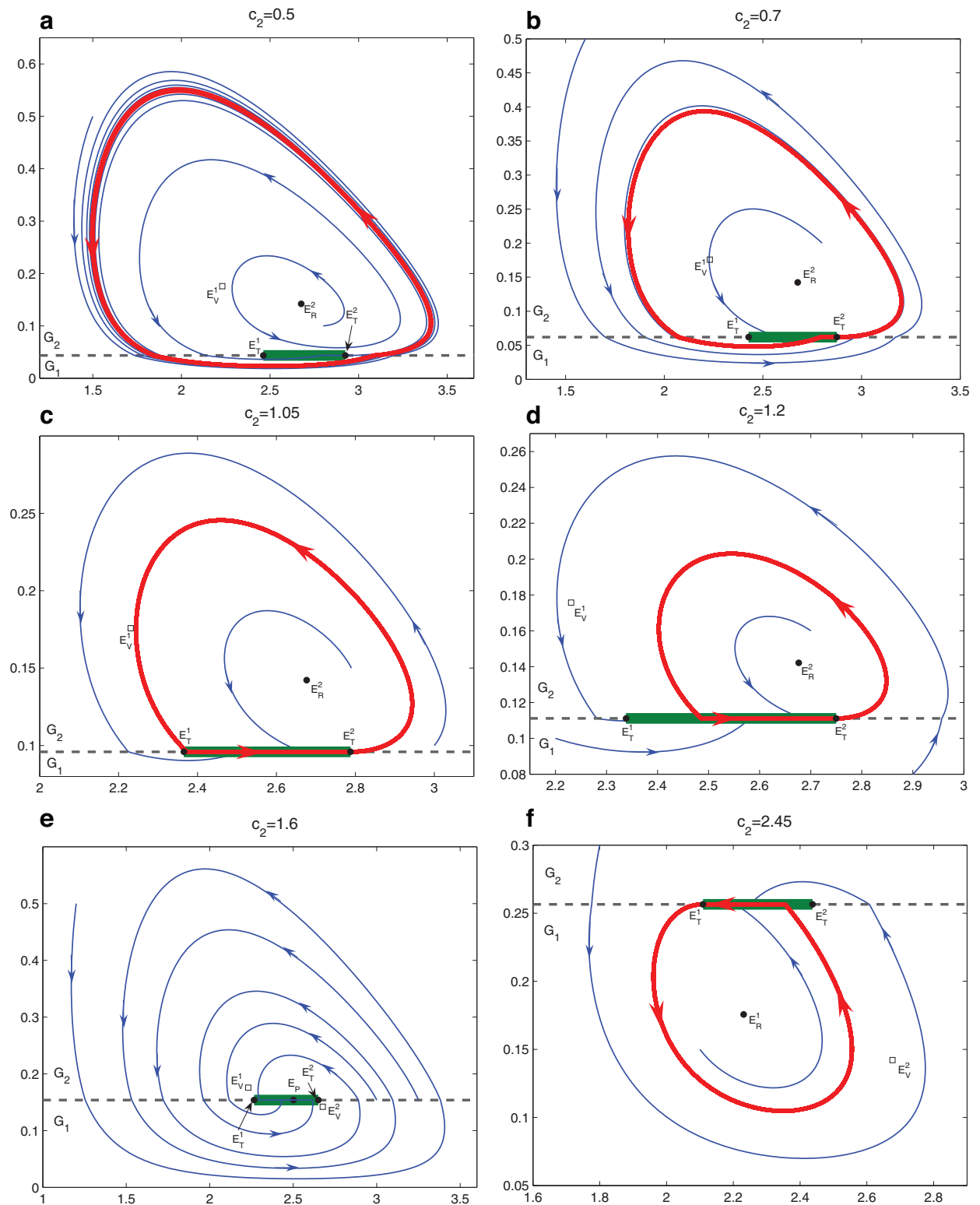


Fig. 5. Buckling and crossing bifurcations for Filippov system (2.10). Parameters are:  $A = 0.9$ ,  $\nu = 0.1$ ,  $\beta = 2$ ,  $b_1 = 1$ ,  $b_2 = 2$ ,  $c_1 = 6$ ,  $k = 0.1$ ,  $\mu = 0.052$ .

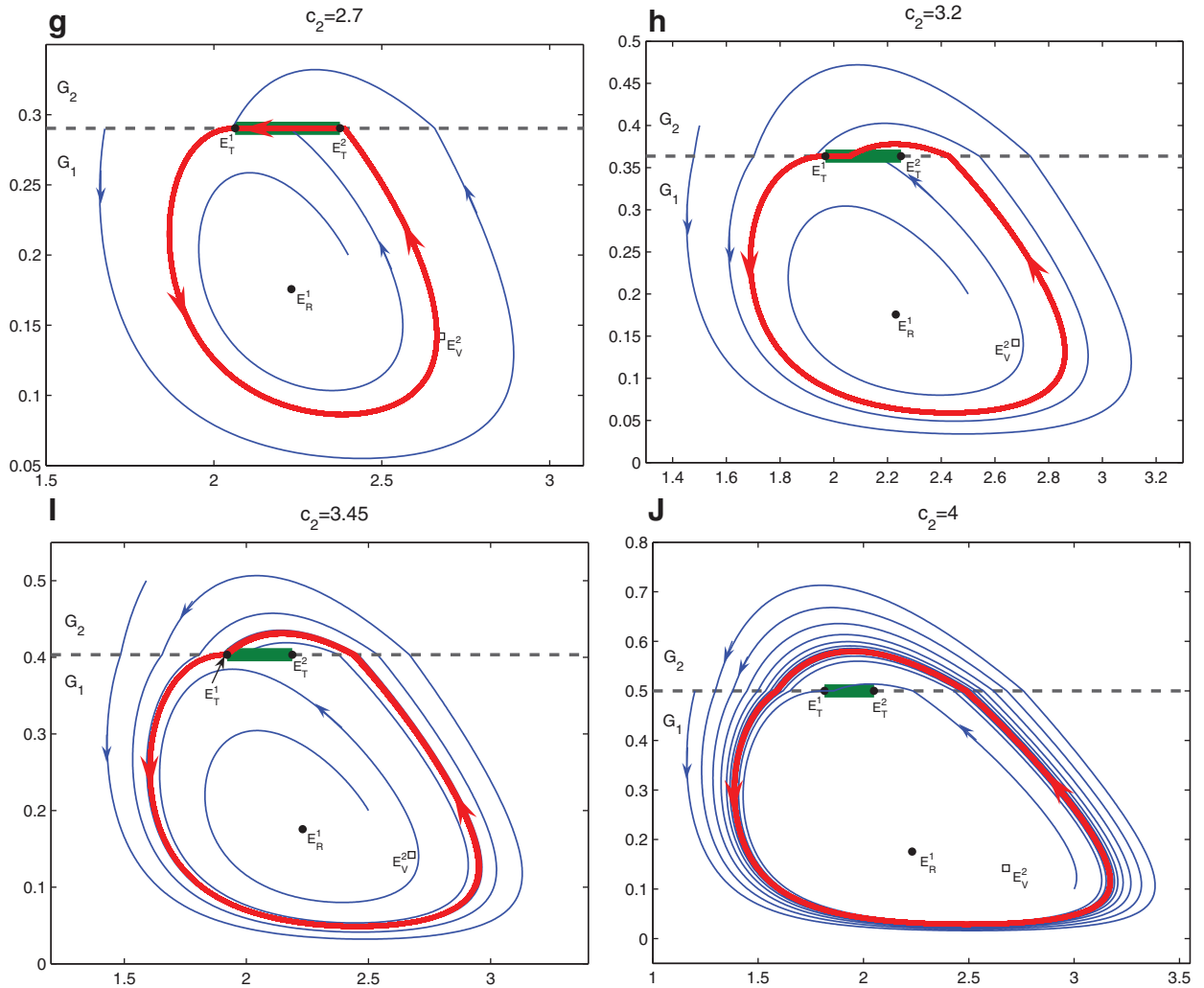


Fig. 5. Continued

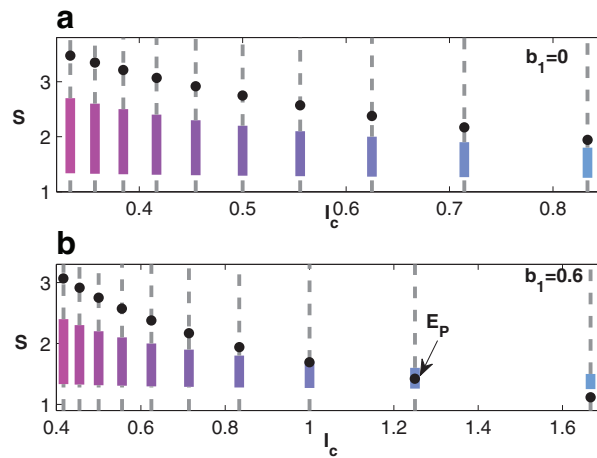
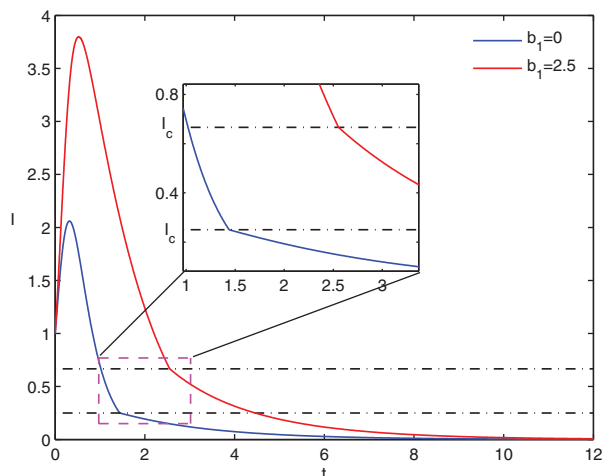
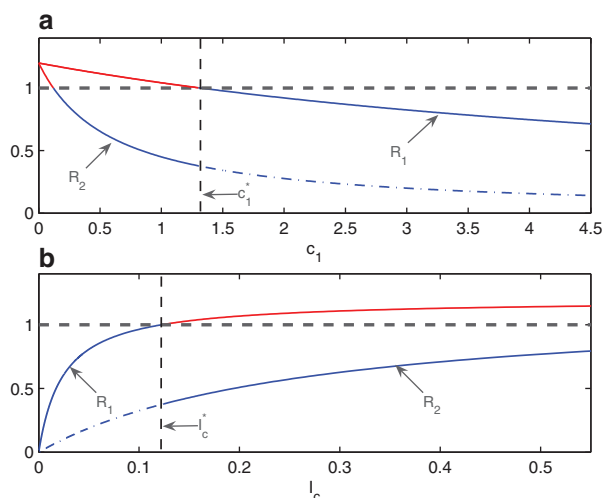


Fig. 6. The impacts of  $b_1$  on the existence of pseudo-equilibrium of Filippov system (2.10). Parameters are:  $A = 2.2$ ,  $\nu = 1.2$ ,  $\beta = 1$ ,  $b_2 = 1$ ,  $c_2 = 0.5$ ,  $k = 10$ ,  $\mu = 0.3$  and  $c_1 = 0.6, 0.7, 0.8, \dots, 1.5$ .



**Fig. 7.** The time series of  $I$  with different parameter value  $b_1$ . Parameters are:  $A = 0.8$ ,  $\mu = 2.2$ ,  $\nu = 0.71$ ,  $\beta = 0.8$ ,  $k = 20$ ,  $b_2 = 2$ ,  $c_2 = 2$ ,  $c_2 = 1$ ,  $b_2 = 2$  and initial value  $(S_0, I_0) = (10, 1)$ . (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)



**Fig. 8.** The monotonicity of  $R_1$  and  $R_2$  with respect to  $c_1$  and  $I_c$ . Parameters are:  $A = 0.6$ ,  $\mu = 0.5$ ,  $\nu = 0.6$ ,  $\beta = 0.6$ ,  $k = 2$ ,  $b_2 = 5$ ,  $c_2 = 0.8$ ,  $b_1 = 0.1$ .

of the peak of patients, and the peak value increases obviously, which indicates that the limited medical resources does not facilitate the treatment of infectious disease.

Therefore, in order to prevent and control the spread of emerging infectious disease, it is crucial to implement the selective strategy timely to control the basic reproduction numbers  $R_i < 1$ ,  $i = 1, 2$ . Then, the key parameters and threshold value  $I_c$  which affect the basic reproduction numbers  $R_i$  and selective strategy are investigated, respectively.

Obviously, it follows from the expressions of the basic reproduction numbers  $R_i$  that they are monotonically decreasing functions with respect to  $c_1$ . Meanwhile, we can solve  $R_{01} = 1$  with respect to  $c_1$  and derive that

$$c_1^* = \frac{(A\beta - \mu\nu)(1 + b_2k)}{\mu}, \quad (6.1)$$

so  $R_{01}(c_1) < 1$  provided  $c_1 > c_1^*$ , as shown in Fig. 8(A). An improved the maximum cure rate for SARS will help prevent and control the spread of SARS.

Furthermore, the threshold value  $I_c$  is a monotonically decreasing function with respect to  $c_1$ . Therefore, in order to control the spread of SARS, it is crucial to reduce the threshold value  $I_c$  to  $I_c^*$ , here

$$I_c^* = \frac{c_2}{c_1^*b_2 - c_2b_1} = \frac{c_2\mu}{b_2(A\beta - \mu\nu)(1 + b_2k) - \mu c_2b_1}. \quad (6.2)$$

That is, the smaller the threshold value  $I_c$  is, the more beneficial to prevent and control SARS, as shown in Fig. 8(B). Those indicate that it is best to timely selective treatment for SARS infected cases so as not to miss the best timing of treatment.



## 7. Concluding remarks

In order to understand the effect of the delayed treating ( $b_1 \neq 0$ ) for the patients with emerging infectious disease on control strategy, we have deduced a non-smooth infectious disease model induced by selection pressures. Analysis of this model reveals rich dynamics including local and global sliding bifurcations. Our main results show that reducing the threshold value to an appropriate level could contribute to the efficacy on prevention and treatment of emerging infectious disease, which indicates that the selection pressures can be beneficial to prevent the emerging infectious disease under medical resource limitation.

Comparing the results for the model with  $b_1 = 0$  we have studied in Ref. [10] with  $b_1 \neq 0$ , we conclude that the term  $b_1$  makes the dynamical behavior of the system change more interesting and complicated. By using theoretical techniques [26,27,31], the existence conditions for sliding segment, sliding mode dynamics and different types of equilibria such as regular equilibrium, pseudo-equilibrium, boundary equilibrium and tangent point have been provided. Further, numerical sliding bifurcation analyses show that the proposed Filippov system has more rich local and global sliding bifurcations than the case studied in Ref. [10], as shown in Figs. 1–5 for more details. Especially, the most interesting results are those for the fixed parameter set as the bifurcation parameter varies, the sliding bifurcations occur sequentially: crossing  $\rightarrow$  buckling  $\rightarrow$  real/virtual equilibrium  $\rightarrow$  buckling  $\rightarrow$  crossing.

According to the analyses of key parameters and biological significance, the results indicate that the dynamic behavior of the Filippov system (2.10) could be dramatically affected by the existence of medical resource limitation (i.e.,  $b_1 \neq 0$ ), see Fig. 6(B) for more detail, the existence and stability of pseudo-equilibrium shows that the emerging infectious disease will become endemic. That is, the case shown in Figs. 4(D) and 5(E), which reveals that there are several hidden factors that can adverse affect the control strategy under medical resource limitation. Therefore, it is very necessary to implement the selective strategy in the control and treatment of SARS, see Fig. 8. Meanwhile, this points to the urgent need for improvement in medical facilities, access to rapid diagnosis and treatment with more effective drugs for SARS.

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