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Mathematical proof: from mathematics to school mathematics

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Proof plays a central role in developing, establishing and communicating mathematical knowledge. Nevertheless, it is not such a central element in school mathematics. This article discusses some issues involving mathematical proof in school, intending to characterize the understanding of mathematical proof in school, its function and the meaning and relevance attributed to the notion of simple proof. The main conclusions suggest that the idea of addressing mathematical proof at all levels of school is a recent idea that is not yet fully implemented in schools. It requires an adaptation of the understanding of proof to the age of the students, reducing the level of formality and allowing the students to experience the different functions of proof and not only the function of verification. Among the different functions of proof, the function of explanation deserves special attention due to the illumination and empowerment that it can bring to the students and their learning. The way this function of proof relates to the notion of simple proof (and the related aesthetic issues) seems relevant enough to make it, in the future, a focus of attention for the teachers who address mathematical proof in the classroom.

This article is part of the theme issue 'The notion of 'simple proof' - Hilbert's 24th problem'.

1. Introduction

> Mathematics and school mathematics.
> Mathematicians and mathematics students.
> What do they have in common? What is it specific to the latter?

In this article, I start from some results of a study on teachers' perceptions about proof [1] and I discuss some issues involving mathematical proof in school, intending to characterize the understanding of mathematical proof in school, its function and the meaning and relevance attributed to the notion of simple proof. The main goal is to analyse some differences and some similarities between mathematics and school mathematics in regard to proof and use them to characterize some issues of the mathematical proof at school level.

In recent years, much research has been done concerning proof and argumentation as illustrated by two chapters recently published which review research in the field. According to them, the research on the area can be organized according to the following topics: focusing on different perspectives of proof; or on students' conceptions and learning and specifically on their conceptions of proof and the proof process, on experts and novices' use of examples, and on knowledge, tasks and tools that promote success in generating proof; or on classroom-based research and specifically on students' processes of argumentation and proof, on teachers' ways of dealing with it, and on interventions aimed at improving it; or on teacher knowledge and development [2,3]. Nevertheless, and besides increasing interest in research on proof at school level, not much attention has been given to the issue of simplicity of proof. In this article, the intention is to focus on the simplicity of proof and discuss the impact on it of the different understandings of proof and its functions in mathematics and in school mathematics.

Hilbert's 24th problem is about the development of a theory that allows choosing between two (or more) different proofs of the same result. The idea is that different proofs are, of course, different, but in this difference, one of the proofs might have some special quality. As I discuss later, some authors speak about simplicity, others refer to a special beauty and some others mention an aesthetic characteristic of proofs.

If in mathematics, it is often possible to find different proofs of the same result, when moving to the context of school mathematics, the diversity is even greater. The mathematical knowledge of a student is of course different from that of a mathematician. The student does not have the same level of familiarity with a formal language and notation, nor the same level of proficiency with algebraic manipulation and so on. As a consequence, in the school context, it is possible to be confronted with proposals of proof like the ones in figures 1 and 2.

Figure 1 describes a proposal of proof developed by a student. The goal is to prove that the sum of two even numbers is still an even number.

Figure 2 presents a geometric representation that intends to proof that the sum of the first n odd numbers is n^2.

A first observation about the proofs presented in these figures is that they differ from the proof we would expect a mathematician to develop. And the unavoidable question is: what is a mathematical proof? And this means considering both the mathematical and the school mathematics context.

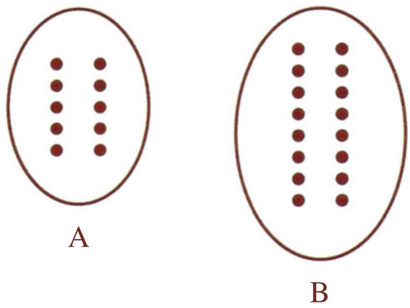
2. Proof: from mathematics to school mathematics

> Proof is a formal demonstration of a result, a sequence of logical arguments that allows establishing the veracity of a mathematical property.
>
> Perspective of one teacher [1, p. 77]

Proof is assumed to be central in mathematics. Tsamir *et al.* [6] refer to it as the heart of mathematics. Blanton & Stylianou [7] emphasize the same idea quoting Schoenfeld [8, p. 12] and his view of proof as the 'soul of mathematics' and Ross [9, p. 2] and his point of view of proof as the 'essence of mathematics'. In turn, Stylianides [10] speaks about the fundamental role proof plays in developing, establishing and communicating mathematical knowledge.

This relevance of proof in mathematics led several researchers to assume its relevance in school mathematics as well [10]. That is the case with Tsamir *et al.* [6], who speak about proof as a key

The student began drawing the following picture:



After that the student said that if a number is even it can be represented by a set of pairs of points such as the ones in A and B. He stated that the number of pairs in A and B is not relevant. And then he said that if we put B below A this will represent the sum of A and B, concluding that as A+B is represented as a set of pairs of points, this number is still an even number.

Figure 1. Description of a proof developed by one student [4]. (Online version in colour.)

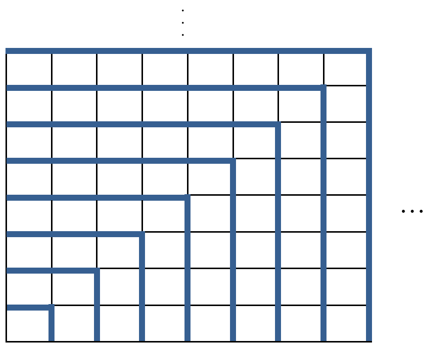


Figure 2. A classroom proof [5]. (Online version in colour.)

component of any mathematical education. Also Dawkins & Weber [11, p. 123] assume proof as 'one of the cornerstones of mathematical practice', recognizing the ability to create proofs as an important goal of mathematics education. As Ko & Knuth [12, p. 68] state, 'proving and refuting are crucial abilities in advanced mathematical thinking because they help demonstrate whether and why propositions are true or false'. And this kind of work is useful to get insight into the mathematical ideas involved, helping the students to achieve a deeper understanding of the concepts and the relations between them. In a mathematical proof, definitions, statements and procedures are intertwined in a suitable way in order to get the desired result. This process improves the students' comprehension of the logic behind the statement [12]. This is also the case with counterexamples and the significant role they play in mathematics. They can elucidate why a conjecture is not true, because one is enough to determine falsity. 'Taken together, mathematical proofs and counterexamples can provide students with insight into meanings behind statements and also help them see why statements are true or false. Accordingly, undergraduate students in advanced mathematics are expected to learn and to use both proofs and counterexamples throughout the undergraduate mathematics curriculum' [12, p. 68].

Stylianides *et al.* [2, p. 315], in their analysis of proof and argumentation in mathematics education research, even assume there is 'a widespread agreement among mathematics

educators on the significance of proof in students' learning of mathematics'. This is a point of view also shared by Stylianides *et al.* [3], who stress the role proof can play in emphasizing the logic of mathematics instead of the authority of the teacher or the textbook.

Nevertheless, as Knuth [13,14] points out, the central role assigned to proof in mathematics and in mathematicians' practice is not present in school mathematics. As the author states, in school, the role of proof has been a peripheral one. 'This absence of proof in school mathematics has not gone unnoted and, in fact, has been a target of criticism' [13, p. 61]. And one of the reasons usually pointed out by critics is that it can give the students a wrong idea about the nature of the mathematics [13,14].

As Schoenfeld [15, p. 76] states, proof is a fundamental part of 'doing, communicating, and recording mathematics' and it is not possible to turn it into a separate part of the mathematics, as usually happens in school. Traditionally, when proof is a reality in school, it includes only students at high school level and studying geometry [13,14,16]. Ellis *et al*. [16] consider that it was only in 2000, when the Principles and Standards [17] were published, that researchers and policy makers began to argue in favour of incorporating proof in mathematics education at all levels. The main reason for this argument, beyond the central role of proof in mathematics, is that it does not seem reasonable to introduce proof only at high school level and to expect that the students develop an immediate appreciation for it [13,14]. However, this argumentation in favour of introducing proof at all school levels does not necessarily mean that proof has actually been integrated into the mathematical experiences of all students, including those of the elementary grades [10]. Actually, as Knuth [13,14] points out, these recommendations are too demanding for teachers, requiring a deep understanding of the nature and role of proof. And even when there is agreement among mathematics educators of the relevance of ascribing to proof a central role, several research studies suggest that turning it into a reality in the classroom is a challenging task for both teachers and researchers [3,11].

The fact is that, 'despite the increased attention and emphasis being placed on proof in school mathematics, students of all ages continue to struggle learning to prove' [18, p. 2]. Some studies suggest that the difficulties are related to the abrupt way in which formal proof is introduced at the secondary level [16], recommending an earlier and less formal introduction. At the earlier levels, the students' experience with proof could start with example-based justifications, and continue using progressively more general and deductive arguments [16,18]. According to these authors, some researchers have proposed hierarchies that try to define ways in which this process can be implemented. Nevertheless, difficulties remain about the best way to foster students' transition from less formal example-based arguments to more general deductive proofs. Some studies suggest that the difficulties in the students' transition are due to an excessive focus on the use of examples as a means of justification [18]. And they recommend an approach where the students have the opportunity to became aware of the limitations of the use of examples, and thus recognize the need for proof. But even in these circumstances, the students continue to face difficulties in learning how to prove, and the teachers tend not to be successful in their mission to help the students learn to prove.

The students have difficulty in constructing the arguments necessary to prove, in understanding the methods of proof and even in understanding what constitutes a proof and what does not [7]. Actually, the notion of what constitutes a mathematical proof has changed over time [19] and is not a matter of consensus [20,21]. Bleiler-Baxter & Pair [22, p. 16] follow de Villiers's [23] point of view and define proof as 'logical deduction that is used to verify, explain, systematize, discover, and communicate mathematics' (including both the proof as a written argument and the act of proving). Steele & Rogers [21, p. 161], inspired by previous work by other authors, define proof as 'a mathematical argument that is general for a class of mathematical ideas and establishes the truth of a mathematical statement based on mathematical facts that are accepted or have been previously proven'. Tall [24, p. 93] notes, 'though proof is the central idea of modern mathematics, quite what it *is* is a matter of implicit, rather than explicit, agreement between members of the mathematics community'. And Smith [19, p. 75] observes that

proof is often assumed to be 'a convincing argument', and that what is accepted as convincing is determined by the experts, the teacher or the textbook.

Dawkins [25, p. 63] refers to terms such as 'formal' and 'rigorous' as being closely related to the meaning of proof; nevertheless these terms are not related to clear definitions of their meanings. And actually, according to Rocha [1], these are terms usually used by teachers to define proof. That is the case of the teachers enrolled on a study, who use these terms to characterize mathematical proof (see the characterization by one of these teachers at the beginning of this section). And the level of formalism is also important when characterizing proof at school level. In the words of the same teacher, the 'concept of proof does not change. However, especially when working with younger students, (. . .) we must consider the level of abstraction and formalism included. It is sometimes better to lose in formalism and gain in intuition' [1, p. 78]. So, although the teacher considers that the concept of proof is the same in mathematics and in school mathematics, if the level of formalism required in school is different, then there is a change in the conceptualization of proof. This difference is recognized by Stylianides & Ball [26, p. 309] in the way in which they describe proof in school mathematics as

> a mathematical argument that fulfils three criteria:
>
> (i) it uses statements accepted by the classroom community (set of accepted statements) that are true and available without further justification;
> (ii) it employs forms of reasoning (modes of argumentation) that are valid and known to, or within the conceptual reach of, the classroom community; and
> (iii) it is communicated with forms of expression (modes of argument representation) that are appropriate and known to, or within the conceptual reach of, the classroom community.
> Thus the definition of proof breaks each mathematical argument into three major components: the set of accepted statements, the modes of argumentation and the modes of argument representation. In describing the characteristics that these three components need to fulfil for an argument to qualify as a proof, the definition seeks to achieve a defensible balance between two (often competing) considerations: mathematics as a discipline and students as mathematical learners. Regarding the consideration of mathematics as a discipline, the definition requires that proofs use true statements, valid modes of argumentation, and appropriate modes of representation. The terms 'true,' 'valid,' and 'appropriate' should be understood in the context of what is typically agreed upon in the field of mathematics, within the domain of particular mathematical theories. Regarding the consideration of students as mathematical learners, the definition requires that proofs depend on what is accepted, known or conceptually accessible to a classroom community at a given time.

This conceptualization of proof in school is recognized by several authors such as Tsamir *et al*. [6] and Komatsu [27], and one of its main characteristics is the balance it establishes between two dimensions: the mathematics and the learning of mathematics. And the immediate conclusion is that proof in mathematics is different than proof in school mathematics. According to this conceptualization, as Komatsu [27, p. 3] emphasizes, 'proofs do not have to be restricted to formal proofs' represented with mathematical symbols. And this means that informal proofs, such as ones using diagrams or generic examples, should be assumed to be admissible, especially at the lower levels of school.

When proving that the sum of two even numbers is an even number, we expect to see the notation $2n$ ($n \in \aleph$) as a representation of an even number. But in school mathematics, proofs do not have to be restricted to formal proofs [27]. And this means that a set of pairs of points, as used by one student in the description presented in figure 1, is a valid representation for an even number. Nevertheless, this student presents two specific sets of pairs of points. And this means the student is working with an example and not presenting a mathematical argument that is general (in the sense described by Steele & Rogers [21]). However, as stated by Ellis *et al*. [16] and

Knuth *et al.* [18], at the earlier levels the students' experience with proof can start with example-based justifications. Besides that, this student seems to be reasoning beyond a concrete example because he states the number of pairs of points considered is irrelevant for the conclusion. Anyway, at what specific level it is possible to accept an example-based approach and at what levels it is possible to accept a less formal representation is an ill-defined point. And a proof such as the one described in figure 1 is a good illustration of the difference between proof in mathematics and in school mathematics.

3. Functions of proof: from mathematics to school mathematics

> The function of a proof is mainly to attest in a rational and logical way a certain issue that we believe to be true. It is basically the rational justification of a belief.
>
> Perspective of one teacher [1, p. 78]

When thinking about mathematical proof, one of the unavoidable questions is related to the function of proof. And several authors suggested a variety of roles for it [13,14]. One of them is to verify that a statement is true. But another is to 'explain why a statement is true, to communicate mathematical knowledge, to discover or create new mathematics, or to systematize statements into an axiomatic system' [13, p. 63]. According to Bleiler-Baxter & Pair [22], for a mathematician, a proof serves to convince or justify that a certain statement is true. But it also helps to increase the understanding of the result and the related concepts. That is why a proof also has the role of explanation. When the proof allows the mathematician to organize the associated ideas, then systematization occurs and a deductive system of axioms, definitions and theorems is created. In the case where something new and unexpected comes up during the construction of a proof, the proof assumes a discovery function where somehow the result and its proof are created at the same time. But proof also assumes a communication role. Mathematicians use proofs to communicate their work and as a way of presenting their ideas. According to Mejía-Ramos [28], what really motivates the mathematicians is their search for a deeper understanding and this is the reason for their rejection of so-called computer-proofs. This kind of proof checks the validity of every single case but does not allow a deeper understanding of the result.

De Villiers [29] states that, traditionally, justifying or becoming convinced about the validity of a conjecture is the main function attributed to proof; Knuth [13,14] even believes this is the only role that most of the teachers attribute to it. In recent decades, this narrow view of proof has been criticized by authors such as Reid [30]. He believes, as does de Villiers [23], that the five roles proof can assume for a mathematician (verification, explanation, systematization, discovery and communication) can also be important to meaningfully engage students in learning proof. But this means that somehow the students have the opportunity to experience each of these roles of proof [23]. And the reality is that usually the students do not even understand the purpose of proof at all. This situation led Bleiber-Baxter & Pair [22] to look for opportunities to engage students in the different roles of proof, using their reflections to identify activities that enhanced their participation in mathematical proof (examples of tasks to promote experience with the different roles of proof can be found in [31]).

Getting conviction about the truth of a result (meaning verification) is the basic role of proof, nevertheless, students are usually asked to prove intuitive results or theorems presented by the teacher, where the verification is not really a question. As a consequence, the role of verification is only experienced superficially by the students [22]. Besides that, the students (as well as the mathematicians) often become convinced about the truth of a result based on some experiences with examples.

In regard to different functions of proof, beside conviction, the importance of experiencing them has been recognized but few studies have tried to explore learning experiences related to them and not much is known about how to structure learning experiences that engage students in the multiple functions of proof in a meaningfully way [22].

As mentioned before, most of the students are easily convinced about the validity of a mathematical result before any proof of it is presented. This is different from what happens with a mathematician who sometimes has an idea but is not sure about its validity [19]. As a consequence, the purpose of proof in school mathematics is different from the purpose in mathematics. 'The role of proof in the mathematics classroom is primarily explanatory; that is, students should ideally view proofs as giving insight into *why* propositions are true or false' [19, p. 75]. As Tsamir *et al*. [6] highlights, more than being able to experience the different functions of proof, the true overall reason why proof is pointed to as an activity that should be a reality for all the students is related to the way in which it can promote mathematical understanding, giving the students a feeling of empowerment that can lead them to find the true essence of mathematics and admire its power.

Beyond the function of proof, several researchers speak about different types of proofs, putting the focus on this global understanding or empowerment that is deeply connected to proof, and Smith [19] makes a good synthesis of some of these distinctions. That is the case with Hanna [32], who distinguishes between 'proofs that explain' and 'proofs that prove'; the case with Tall [33], who speaks about 'logical' and 'meaningful' proofs; the case with Weber & Alcock [34], who distinguish between 'syntactic' and 'semantic' proofs; and the case with Raman's [35], who focuses on the kind of ideas used in the proof and considers 'heuristic ideas' and 'procedural ideas,' with the concept of a 'key idea' linking the two. The points of view of these authors are slightly different, but as Smith [19, p. 75] emphasizes, what all of them seem to be addressing is two distinct approaches to mathematical proof: 'a procedural, logical approach on which the prover's intuition is not necessarily engaged, and an approach relying on the prover's intuitive understanding of the mathematical structure involved'. It seems desirable that these two types of proof be mastered by students, but one of the types seems to be closely related to the achievement of a deeper understanding and mathematical empowerment. The point that remains open, as Smith [19, p. 75] highlights, is how schools can promote the student's learning of mathematical proof so they can 'develop the sort of mature conception of and facility with proof demonstrated by professional mathematicians'. According to the NCTM [17], the students are expected to be able to actually develop a mathematical proof of a result by the end of grade 12, but the research shows that this goal is not achieved and 'many students ultimately do not succeed in developing an appreciation for mathematical proof' [19, p. 75].

A relevant point is the experience with proof that students live in school, and this is closely related to the teachers and their approaches to proof. As mentioned before, this is not an easy task for them either. And the truth is that so far the research has not provided enough detail on useful options a teacher can use to achieve the desired result [16]. Although the main results suggest that the central function of proof in school should be on improving mathematical understanding, the validity of a proof is assumed as a basic requirement (and this takes us back to the previous point and to the understanding about what a proof is). Nevertheless, Smith [19] suggests that, when analysing a proof, teachers tend to attend to some criteria that do not always include validity. That was the case of some secondary teachers who accepted false proofs as correct ones in cases where some correct algebraic manipulations were included. Other factors, such as the inclusion of some familiar proof technique or the amount of detail presented, were also used as more determinant to accept a proof than the actual correctness of the argument presented [13,14]. Knuth [13,14] also noted that many teachers do not value the activity of proof, assuming that it is not suitable for most of their students. Moreover, the author emphasizes that teachers tended to view proof in a pedagogically limited way, tending to address it like a specific content to study and not addressing relevant functions of proof, such as its communication or explanation function.

The value ascribed to the use of algebraic manipulations and to the use of some proof technique, as emphasized by Smith [19] can lead some teachers to expect that the task of proving that the sum of the first n odd numbers is n^2, is translated into the equality $\sum_{k=1}^{n} 2k - 1 = n^2$ and then proved using mathematical induction. This takes us back to the previous section of this article and to the discussion about whether what is presented in figure 2 can be considered as a proof. I am not going to discuss that here, I am simply going to assume that what is presented in

figure 2 is a proof (at least in the school mathematics context). So we have two different proofs for the same result: one presented in figure 2 and the other using induction. The first one relies on a more intuitive approach [19], being a 'proof that explains' [32] or a 'meaningful proof' [33]. It is easy to start from the square on the left corner representing 1 (the first odd number), see each layer representing the next odd number (3, 5, 7, ...), and then see the resulting square with a side corresponding to the number of odd numbers considered (i.e. n^2). So this is a proof that explains and gives the reader a meaningful idea about the result. And this should be the main role of proof in the mathematics classroom [19].

The second proof, the one based on induction, is a procedural proof, where the prover's intuition is not required [19], being a 'proof that proves' [32] or a 'logical proof' [33]. This proof consists in the performance of a mechanical procedure that brings nothing new to the prover (besides ensuring that the result is valid).

4. The notion of simple proof: from mathematics to school mathematics

> We cannot expect a 7th grade student to use, for example, the proof by absurdity in the same way that a student will do in the 10th grade or even in the 12th grade. If we can expect that a student in the 12th grade, with a normal development in Mathematics, will understand, for example, the proof that an irrational number cannot be written as the quotient of two integers, for a 7th grade student such a proof may be shortened either at the level of the language used and at the level of the arguments used. Whether for some or for others the proof should and has to be simple. Otherwise the students will get lost and would not be able to realize the reason for such a task. Often it is necessary to work on some examples, before moving on to the proof. The examples appeal to the students' intuition and 'pull' them, so they can gain motivation and develop intuition to understand the proof.
>
> Perspective of one teacher [1, p. 75]

Human activity is the subject of appraisals and judgements, and mathematical proof is not different from other activities. Mathematicians analyse each other's proofs and even award them prizes based on such assessments [36]. According to Tsamir *et al.* [6], the first proof of a result achieved by a mathematician is usually complicated. Much more complicated than what it is necessary. It is only later that it is possible to understand that the same result can be achieved in a simpler way. And usually some time after, the same or other mathematicians develop a simpler or more elegant proof. Simplicity and elegance are criteria for good mathematical proof and minimality is an aspect of these [6]. Validity and applicability are not usual criteria for the appreciation of proofs. On the contrary, the usual criteria include references to 'beautiful', 'deep', 'insightful' and 'interesting' [36, p. 87]. This means that aesthetic criteria are the relevant ones. It is the aesthetic that plays the most important role in the process of mathematical thinking [37], the criteria that guide the development and appreciation of mathematics [38]. Nevertheless, as the author points out, aesthetic does not seem to play a relevant role in mathematics education. In the words of Papert [39, p. 192], 'if mathematical aesthetics gets any attention in the schools, it is as an epiphenomenon, an icing on the mathematical cake, rather than as the driving force which makes mathematical thinking function'.

Actually, an analysis of the (national or regional) syllabus of some countries (such as Brazil, France, Mexico, Peru, Portugal or Scotland), at the upper secondary level, shows that these documents do not include any kind of clue to help the teacher decide between two possible proofs of a result. It is possible to find references to the different types of proofs a student should experience or even to theorems which have a proof beyond the students' level of understanding, but nothing else. The Portuguese syllabus, for instance, suggests that 'the students should present mathematical proofs as rigorous as possible' [40, p. 6], but even the meaning of this level of rigour is not clarified. The idea that simplicity is a very relevant issue when considering proof at the school mathematics seems to be shared even by the teachers. That is the case of one teacher who

relates the simplicity of proof to the level of the students in a very clear way (see the beginning of this section for a quotation of this teacher) [1].

Actually, if we think about the understanding of proof at the school level, the simplicity of proof somehow becomes a central issue. As I mentioned before, proofs do not have to be restricted to formal proofs, and this means that the simplicity of the language used has to be present. Also, the modes of argumentation need to be appropriate to the level of the students, and once again this means some simplicity is required. Besides that, we have to attend to the students' knowledge and consider only proofs involving statements known by the students. Once again this puts a focus on simplicity. As a result, the difference between mathematical proof and school mathematical proof turns simplicity of proof into a relevant issue at the school level.

This idea of simplicity in proof at school level can also be related to the idea of minimal proof, in the sense of a proof that is clean, without additional passages and, as such, easier to understand. So, minimal proofs can be desirable in school mathematics ... or not. A proof that does more than prove, that also increases understanding about the relations among the involved concepts [6], possibly is not a minimal proof, but it is a useful proof in the sense that it increases students' understanding in the field being studied. So depending on the function of the proof, minimal can be a good or not so good characteristic of a proof in school mathematics. But Tsamir *et al.* [6] speak of simplicity in the sense of illumination. And this includes in one both the minimal and simple proof that illuminates understanding of the mathematical content. So it is a kind of elegance, a term that according to Inglis & Aberdein [36] several mathematicians assume to be synonymous with simplicity. However, the term that is usually associated with proof simplicity is beauty, and beauty is even recognized as the classic view of simplicity and brevity [36,41]. And according to McAllister [41, p. 22] 'the most important determinant of a proof's perceived beauty is thus the degree to which it lends itself being grasped in a single act of mental apprehension'. But McAllister is speaking from a mathematician's point of view. Thinking of school mathematics and being aware of the difficulties faced by the students when confronted with proofs, this single act of mental apprehension might look more difficult to achieve.

The way in which a mathematical argument starts from the available information and gets to a result determines what Dreyfus & Eisenberg [37] call the aesthetics rating. And the authors point to several properties of this path that enter into play. That is the case of 'its level of prerequisite knowledge, its clarity, its simplicity, its length, its conciseness, its structure, its power, its cleverness, and whether it contains elements of surprise' (p. 3). An argument that requires a lot, in terms of prerequisite knowledge, is an argument that is losing its elegance. No one feels attracted by complicated ideas. A simple and clear argument is always easier to follow than a complicated one. As so, Dreyfus & Eisenberg [37] present brevity as an attractive quality, but they emphasize that brevity is not necessarily number of pages. Instead, the central aspect of brevity 'is the number of logical steps and the step-size' (p. 3).

Somehow, this aesthetic characteristic of proofs seems to be closely related to the functions of proof. As so, it is not surprising that Tsamir *et al.* [6] presents Bell's [42] three characteristics of proofs as a useful tool to analyse mathematical proofs at school level. The first of these characteristics is *verification* and is focused on the truth of the result. The second characteristic is *illumination*, sharing the idea of insight and understanding of the reasons why the result is valid. This is a characteristic closely related to the function of explanation. And the author claims that this characteristic does not impact the validity of the result, 'but its presence in a proof is aesthetically pleasing' [42, p. 24]. And the last characteristic is *systematization*, focused on the organization of mathematical ideas and also one of the important functions of proof.

So an appreciation of a proof includes an analysis of its correctness or incorrectness, but also an appreciation of its level of illumination, beauty, elegance or simplicity [6]. Sometimes an aesthetic appreciation becomes even dominant and is used to decide over the validity of a proof. Stylianides *et al.* [43] refer to some studies where teachers or students base themselves on criteria such as length, explanatory power and visual appearance (if it looks, or not, like a proof) to make a decision.

On the case of the proof presented on figure 2, the formal language usually associated with a mathematical proof is not present. Nevertheless, this proof offers a visual understanding about the reason why the result is valid in a way that an induction approach cannot. This proof allows the verification of the result, but it also provides illumination. It is a simple proof, at the school level, where the representation of successive odd numbers is easy to understand and, at the same time, to see how the result of its sum is the square of side equal to the number of odd numbers, i.e. n^2. This is a beautiful, deep, insightful and interesting proof . . . however, this might not even be assumed as a true proof by a mathematician due to the differences on the notion of proof and its functions in mathematics and in school mathematics.

5. Conclusion

The central role of proof in mathematics is undeniable. Nevertheless, the role of proof in school mathematics is almost a residual one. Only after 2000, some recommendations for the integration of proof in the curriculum at all school levels began to emerge. Prior to that date, the proof was only approached at the end of the secondary school and usually in the study of geometry. Nowadays some curriculums have begun to include references to proof, but the reality in schools is still far from what is prescribed in such documents. The idea of involving students of all ages in proof activities implies adapting proof to the age of the students. As a consequence, school mathematics proof is somehow different from mathematics proof.

In the early school years, mathematical proof is more intuitive and less formal. But even for the students who experience some kind of mathematical proof early, the transition to a more formal approach is usually difficult. Students struggle to understand what is a proof, to construct a proof and even to understand the point of proving. Mathematical proof can have several functions, including verification, explanation, systematization, discovery and communication. All of these functions are important for mathematicians, but school students usually contact only with the verification function. And the circumstances where this contact happens tend to be limited to intuitive results or theorems presented by the teacher, where the verification is not really a question. As a consequence, students do not understand the point of proving.

The research on proof suggests that the students should engage in tasks where they can experience the different functions of proof. Nevertheless, it also suggests that, contrary to what happens in mathematics, in school mathematics, the explanation function is the most relevant one. It is this function that allows a deeper understanding of the concepts, a kind of insight that can be very important for students learning about mathematics. And this function seems to be related to the notion of simple proof.

The proofs presented in figures 1 and 2 differ from the traditional proof we expect a mathematician to produce, mainly because of the notation used. Those are simple proofs at the school level that differ from the simple proofs of a mathematician as a consequence of the difference about the understanding of proof in mathematics and in school mathematics. In mathematics, aesthetics is the main characteristic used to identify simple proofs and although in school mathematics simplicity (especially of the notation used) seems to be the most relevant criteria, aesthetic and the related illumination criteria can also have a relevant role to play, as the proof in figure 2 illustrates.

The idea of a special quality, that allows the distinctions of one particular proof from all the others, something aesthetic that feels illuminating, beautiful, deep, insightful or just simple, seems to be natural to humans. However, the difficulties usually faced by the students in relation to proof might make difficult the appearance of this feeling of illumination. But it decidedly does not make it unwanted. Nevertheless, from an aesthetic point of view, there might be some differences between mathematicians and school mathematics students, but there is always some aesthetic sense present and playing an important role . . . although the notion of simple proof seems to be different at school level and not very valued by the teachers. The way in which the notion of simple proof relates to mathematical empowerment seems relevant enough to suggest turning

it, in the future, into a focus of attention for the teachers who address mathematical proof in the classroom.

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