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Existence and asymptotic behavior of solutions for nonlinear Schrödinger-Poisson systems with steep potential well

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In this paper, we are concerned with a class of Schrödinger-Poisson systems with the asymptotically linear or asymptotically 3-linear nonlinearity. Under some suitable assumptions on V, K, a, and f, we prove the existence, nonexistence, and asymptotic behavior of solutions via variational methods. In particular, the potential V is allowed to be sign-changing for the asymptotically linear case. © *2016 AIP Publishing LLC.* [http://dx.doi.org/10.1063/1.4941036]

I. INTRODUCTION AND MAIN RESULTS

This paper deals with the existence, nonexistence, and asymptotic behavior of solutions to the following nonlinear Schrödinger-Poisson system:

$$(\mathcal{SP})_\lambda \qquad \begin{cases} -\Delta u + \lambda V(x)u + K(x)\phi u = a(x)f(u), & x \in \mathbb{R}^3, \\ -\Delta \phi = K(x)u^2, & x \in \mathbb{R}^3, \end{cases}$$

where $\lambda > 0$ is a parameter, $K(x) \in L^2(\mathbb{R}^3) \cup L^\infty(\mathbb{R}^3)$, $a(x)$ is a positive bounded function, $f(s)$ is either asymptotically linear or asymptotically 3-linear in s at infinity, and the potential V satisfies the following conditions:

(V_1) $V \in C(\mathbb{R}^3, \mathbb{R})$ and V is bounded from below;
(V_2) there exists $b > 0$ such that the set $\{x \in \mathbb{R}^3 : V(x) < b\}$ is nonempty and has finite measure;
(V_3) $\Omega = int\ V^{-1}(0)$ is nonempty and has smooth boundary and $\overline{\Omega} = V^{-1}(0)$.

The kind of hypotheses was first introduced by Bartsch and Wang[6] in the study of a nonlinear Schrödinger equation and has been attracting much attention, see, e.g., Refs. 4, 5, 30, and 36. The conditions (V_1)–(V_3) imply that λV represents a potential well whose depth is controlled by λ. λV is referred as the steep potential well if λ is sufficiently large and one expects to find solutions which localize near its bottom Ω.

Such a system, also known as Schrödinger-Maxwell system, arises in many fields of physics. For example, Schrödinger-Poisson system can describe the interaction of a charged particle with its own electrostatic field in quantum mechanics. The unknowns u and ϕ represent the wave functions associated with the particle and electric potential, and the function V and K are, respectively, an external potential and nonnegative density charge. We refer to Benci and Fortunato[7] and references therein for more details. This model can also appear in semiconductor theory to describe solitary waves.[24,29]

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In recent years, the following Schrödinger-Poisson system,

$$\begin{cases} -\Delta u + V(x)u + K(x)\phi u = f(x,u), & x \in \mathbb{R}^3, \\ -\Delta \phi = K(x)u^2, & x \in \mathbb{R}^3, \end{cases}$$

has been widely studied under variant assumptions on V, K, and f via variational methods, and existence, nonexistence and multiplicity results are obtained in many papers, see, e.g., Refs. 2, 3, 9–11, 18, 20, 21, 25, 28, and 40–43.

Recently, Wang and Zhou[35] studied the following asymptotically linear Schrödinger-Poisson system:

$$\begin{cases} -\Delta u + V(x)u + \lambda\phi u = f(x,u), & x \in \mathbb{R}^3, \\ -\Delta \phi = u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $\lambda > 0$ is a parameter, and $V \not\equiv const$. They proved that problem (1.1) has a positive solution for λ small and has no any nontrivial solution for λ large. Later, Zhu[44] also studied system (1.1) with $V(x) \equiv \beta$, and asymptotically linear nonlinearity $f(x,s)s$ where $f(x,s)$ tends to $p(x)$ and $q(x) \in L^\infty(\mathbb{R}^3)$, respectively, as $s \to 0$ and $s \to +\infty$. The author obtained the existence and nonexistence results, depending on the parameters β and λ.

Very recently, Cerami and Vaira[8] considered the following system:

$$\begin{cases} -\Delta u + u + K(x)\phi u = a(x)|u|^{p-1}u, & x \in \mathbb{R}^3, \\ -\Delta \phi = K(x)u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.2}$$

where $3 < p < 5$, and $K \in L^2(\mathbb{R}^3)$. They proved that (1.2) possesses a positive ground state solution by minimization on Nehari manifold when $a : \mathbb{R}^3 \to \mathbb{R}$ and $K : \mathbb{R}^3 \to \mathbb{R}$ are nonnegative functions such that

$$\lim_{|x|\to\infty} a(x) = a_\infty, \quad \lim_{|x|\to\infty} K(x) = 0.$$

Later, Sun *et al.*[31] also studied system (1.2) with asymptotically linear nonlinearity $a(x)f(u)$. They obtained the existence of positive ground state solutions under suitable $K \in L^2(\mathbb{R}^3)$ and $a(x)$ by the mountain pass theorem.

Nearly, Jiang and Zhou[19] first applied the steep potential well into Schrödinger-Poisson system and proved the existence of solutions. Moreover, they also studied the asymptotic behavior of solutions by combining domains approximation with priori estimates. Subsequently, Zhao *et al.*[42] considered system $(\mathcal{SP})_\lambda$ with V satisfying (V_1)–(V_3) and $a(x)f(u) = |u|^{p-2}u$, where $p \in (3,6)$. By using variational setting of Refs. 13 and 14, they obtained the existence and asymptotic behavior of nontrivial solutions. In particular, the potential V is allowed to be sign-changing for the case $p \in (4,6)$. Later, Sun and Wu[32] studied the following Kirchhoff type problem:

$$\begin{cases} -\Big(a + b\displaystyle\int_{\mathbb{R}^N} |\nabla u|^2\Big)\Delta u + \lambda V(x)u = f(x,u), & x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases}$$

where $N \geq 3$, the parameters $a, b, \lambda > 0$, the potential V satisfies the conditions (V_1)–(V_3) and $V \geq 0$, and the nonlinearity $f(x,s)$ is asymptotically k-linear $(k = 1,3,4)$ with respect to s at infinity. They proved the existence and nonexistence of nontrivial solutions by using variational methods. Furthermore, they also explored the asymptotic behavior of solutions.

Moreover, the semiclassical limit of system (1.1) was also studied recently. More precisely, the first equation of system (1.1) looks like $-\epsilon^2\Delta u + V(x)u + K(x)\phi u = a(x)|u|^{p-1}u$, and the solutions exhibit concentration phenomena as the parameter ϵ goes to zero. We refer to Refs. 1, 15–17, 27, 33, 34, and 37 and the references therein on this subject.

Motivated by the works described above, particularly, by the results in Ref. 42, the aim of this paper is to consider system $(\mathcal{SP})_\lambda$ with steep potential well. To our best knowledge, this case has not ever been studied for the asymptotically linear nonlinear term. We mainly study the existence and nonexistence of solutions for system $(\mathcal{SP})_\lambda$ with the asymptotically linear nonlinearity via

variational methods. Moreover, the existence and asymptotic behavior of ground state solutions are also discussed for system $(\mathcal{SP})_\lambda$ with the asymptotically 3-linear nonlinearity.

If $f(s)$ is asymptotically linear in s at infinity, we assume that

(F_1) $f \in C(\mathbb{R}^3,\mathbb{R}^+)$, $f(s) \equiv 0$ for all $s \le 0$ and $f(s)/s \to 0$ as $s \to 0$;
(F_2) there exists $l \in (0,+\infty)$ such that $f(s)/s \to l$ as $s \to +\infty$.
(A_1) $a(x) \in L^\infty(\mathbb{R}^3)$, $a(x) \ge a_0 > 0$ for all $x \in \mathbb{R}^3$, and there exists $R_0 > 0$ such that

$$\sup\{f(s)/s : s > 0\} < \inf\{V(x)/a(x) : |x| \ge R_0 \text{ and } V(x) \ge b\}; \tag{1.3}$$

(A_2) $a(x) \in L^\infty(\mathbb{R}^3)$, $a(x) \ge a_0 > 0$ for all $x \in \mathbb{R}^3$.

Remark 1.1. In Ref. 23, the authors first assumed

$$\sup\{f(s)/s : s > 0\} < \inf\{V(x)/a(x) : |x| \ge R_0\}. \tag{1.4}$$

This condition plays an important role in verifying the Cerami condition ((C) condition for short). In view of (1.4), it is easy to see that the positivity of right-hand side is necessary. Nevertheless, from (V_1) and (V_2), we deduce that $\liminf_{|x|\to\infty} V(x) = 0$ is allowed. So, we cannot make use of condition (1.4) as Ref. 23. Fortunately, by (V_2), we find condition (1.3) to overcome this difficulty. Moreover, it is quite obvious that condition (1.3) is weaker than condition (1.4).

Before stating our main results, we need to introduce some notations.

Notation 1.1. The letters C, C_i denote different positive constants whose exact value is inessential. The usual norm in Lebesgue space $L^s(\mathbb{R}^3)$ with $1 \le s \le +\infty$ is denoted by $|\cdot|_s$. $H^1(\mathbb{R}^3)$ and $D^{1,2}(\mathbb{R}^3)$ denote the usual Sobolev space, and S is the best Sobolev constant for the Sobolev embedding of $D^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$. For any $r > 0$ and $y \in \mathbb{R}^3$, $B_r(y)$ denotes the ball of radius r centered at y, and $|B_r(y)|$ denotes its Lebesgue measure.

Our main results are as follows.

Theorem 1.1. *Suppose that $V(x) < 0$ for some x, (V_1), (V_2), (F_1), (F_2) and (A_1) are satisfied. If $K(x) > 0$ for $x \in \mathbb{R}^3$ and $K \in L^2(\mathbb{R}^3)$ (or $K(x) \ge k_0 > 0$ for $x \in \mathbb{R}^3$ and $K \in L^\infty(\mathbb{R}^3)$), then there exist $\lambda_m \xrightarrow{m} \infty$ and $k^*_{\lambda_m} > 0$ such that system $(\mathcal{SP})_\lambda$ has a nontrivial solution for each $\lambda = \lambda_m$ and $|K|_2 < k^*_{\lambda_m}$ $\left(\text{or } |K|_\infty < k^*_{\lambda_m}\right)$.*

If $V \ge 0$, we need to consider the following minimization problem:

$$\mu^* = \inf\left\{\int_\Omega |\nabla u|^2 dx : u \in H^1_0(\Omega), \int_\Omega a(x)u^2 dx = 1\right\}. \tag{1.5}$$

Using (A_2), we see that $\mu^* \in (0,+\infty)$ and it can be achieved. Then, we have the following results.

Theorem 1.2. *Suppose that $V \ge 0$, (V_1)–(V_3), (F_1), (F_2), and (A_1) are satisfied. Let $l > \mu^*$, where μ^* is given in (1.5). If $K(x) > 0$ for $x \in \mathbb{R}^3$ and $K \in L^2(\mathbb{R}^3)$ (or $K(x) \ge k_0 > 0$ for $x \in \mathbb{R}^3$ and $K \in L^\infty(\mathbb{R}^3)$),then there exists $k^* > 0$ such that system $(\mathcal{SP})_\lambda$ has a positive ground state solution (or positive solution) for all $\lambda \ge 1$ and $|K|_2 < k^*$ (or $|K|_\infty < k^*$).*

Example 1.1. It is not difficult to find some functions a and f such that the conditions (F_1), (F_2), and (A_1) are satisfied. For example, for any fixed $l > 0$, set

$$f(s) = \begin{cases} \dfrac{ls^2}{1+s}, & \text{if } s > 0, \\ 0, & \text{if } s \le 0. \end{cases}$$

It is easy to see that the function f satisfies the conditions (F_1) and (F_2). Moreover, we yield that $\sup\{f(s)/s : s > 0\} = l$. Let $a(x) \equiv 1$ for all $x \in \mathbb{R}^3$. Taking $l \in (0,b)$, we obtain that (1.3) holds. As a consequence, the condition (A_1) holds. By (1.5), we see that $\mu^* = \lambda_1(\Omega)$, where $\lambda_1(\Omega)$ is the first eigenvalue of $-\Delta$ in $H^1_0(\Omega)$. Therefore, if we choose $l \in (\lambda_1(\Omega),b)$, then the conditions (F_1), (F_2), and (A_1) are satisfied and $l > \mu^*$.

Theorem 1.3. *Suppose that $V \geq 0$, (V_1), (V_2), (F_1), (F_2), and (A_2) are satisfied. If $K(x) \geq k_0 > 0$ for $x \in \mathbb{R}^3$ and $K \in L^\infty(\mathbb{R}^3)$, then there exists $k_* > 0$ such that system $(\mathcal{SP})_\lambda$ has no nontrivial solution for all $\lambda \geq 1$ and $k_0 \geq k_*$.*

Remark 1.2. Note that Theorem 1.3 remains true if we replace the condition (A_2) by

(A_3) $a(x) \in L^\infty(\mathbb{R}^3)$, $a(x) \geq (\not\equiv) 0$ *for all* $x \in \mathbb{R}^3$.

If $f(s)$ is asymptotically 3-linear in s at infinity, we assume that

(F_3) there exists $l \in (0, +\infty)$ such that $f(s)/s^3 \to l$ as $s \to +\infty$;
(F_4) $f(s)s - 4F(s) \geq 0$ for all $s > 0$, where $F(s) = \int_0^s f(t)dt$.

In the following, we give some examples, in which our conditions (F_1), (F_3), and (F_4) are satisfied.

Example 1.2. Let $f(s) = (s^+)^3$, we see that the conditions (F_1), (F_3), and (F_4) are satisfied. Another simple is that $f(s) = (s^+)^3 - (s^+)^{p-1}$ with $2 \leq p < 4$.

Theorem 1.4. *Suppose that $V \geq 0$, (V_1)–(V_3), (F_1), (F_3), (F_4), and (A_2) are satisfied. If $K(x) \geq 0$ for $x \in \mathbb{R}^3$, $K \in L^2(\mathbb{R}^3) \cap L^\infty_{loc}(\mathbb{R}^3)$ $\big($or $K \in L^\infty(\mathbb{R}^3)\big)$, then there exist $\Lambda > 0$ such that system $(\mathcal{SP})_\lambda$ has a positive ground state solution for all $\lambda \geq \Lambda$.*

Remark 1.3. Compared with Theorem 1.2 of Ref. 42, we find a positive ground state solution in Theorem 1.4, while in Ref. 42 the authors obtained a nontrivial solution.

On the asymptotic behavior of positive ground state solutions, we have the following result.

Theorem 1.5. *Let $(u_\lambda, \phi_{u_\lambda})$ be the positive ground state solutions obtained in Theorem 1.4. Then $u_\lambda \to u_0$ in $H^1(\mathbb{R}^3)$, $\phi_{u_\lambda} \to \phi_{u_0}$ in $D^{1,2}(\mathbb{R}^3)$ as $\lambda \to \infty$, where $\phi_{u_0}(x) = \int_\Omega \frac{K(y)u^2(y)}{4\pi|x-y|}dy$ and $u_0 \in H_0^1(\Omega)$ is a positive solution of*

$$(\mathcal{SP})_\infty \qquad \begin{cases} -\Delta u + K(x)\left(\displaystyle\int_\Omega \frac{K(y)u^2(y)}{4\pi|x-y|}dy\right)u = a(x)f(u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases}$$

The proof is based on variational methods. By comparing with the previous works, the main difficulty in proving our theorems is the lack of compactness. The competing effect of the nonlocal term with the nonlinearity f and the lack of compactness of the Sobolev embedding of $H^1(\mathbb{R}^3)$ into $L^p(\mathbb{R}^3)$, $p \in (2, 6)$ prevent us from using the variational methods in a standard way. Since we do not assume the condition that the potential V is radially symmetric, or the hypothesis used in Refs. 3 and 43 (that is, $V(\infty) = \lim_{|y| \to +\infty} V(y) \geq V(x)$ for a.e. $x \in \mathbb{R}^3$, and the strict inequality holds on a positive measure set), we cannot make use of the usual way to recover compactness, for instance, restricting in the subspace of $H^1(\mathbb{R}^3)$ consisting of radially symmetric functions or using concentration compactness methods. In order to recover the compactness, we establish two compactness results. More precisely, when we prove Theorems 1.1 and 1.2, we borrow some ideas used in Refs. 12 and 35 and establish the first compactness result: $\int_{|x| \geq R}(|\nabla u_n|^2 + \lambda V^+(x)u_n^2)dx \leq \epsilon$; when we prove Theorem 1.4, we borrow some ideas used in Refs. 4 and 13 and establish the parameter dependent compactness conditions which is the second compactness result. On the other hand, if $f(s)$ is asymptotically linear with respect to s at infinity, it is known that (AR) condition on f is no longer available. It gives rise to some difficulties for verifying the linking geometry, mountain pass geometry, and boundedness of a (C) sequence. Inspired by Refs. 28 and 31, we overcome these difficulties by a very simple and clear way.

This paper is organized as follows. In Section II, we give the variational framework for system $(\mathcal{SP})_\lambda$ and establish the compactness conditions. Sections III and IV are devoted to the proof of our main results.

II. VARIATIONAL SETTING AND COMPACTNESS CONDITION

In this section, we give the variational setting for system $(\mathcal{SP})_\lambda$ and establish the compactness conditions.

Let $V(x) = V^+(x) - V^-(x)$, where $V^\pm(x) = \max\{\pm V(x), 0\}$. Let

$$E = \left\{ u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} V^+(x)u^2 < \infty \right\}$$

be equipped with the inner product and norm

$$\langle u, v\rangle = \int_{\mathbb{R}^3} \left(\nabla u \nabla v + V^+(x)uv\right), \quad \|u\| = \langle u, u\rangle^{1/2}.$$

For $\lambda > 0$, we shall also need the following inner product and norm

$$\langle u, v\rangle_\lambda = \int_{\mathbb{R}^3} \left(\nabla u \nabla v + \lambda V^+(x)uv\right), \qquad \|u\|_\lambda = \langle u, u\rangle_\lambda^{1/2}.$$

It is clear that $\|u\| \le \|u\|_\lambda$ for $\lambda \ge 1$. Set $E_\lambda = (E, \|\cdot\|_\lambda)$. It follows from the conditions (V_1), (V_2) and the Hölder and Sobolev inequalities that

$$\begin{aligned}
\int_{\mathbb{R}^3} \left(|\nabla u|^2 + u^2\right) dx &= \int_{\mathbb{R}^3} |\nabla u|^2 dx + \int_{\{V<b\}} u^2 dx + \int_{\{V \ge b\}} u^2 dx \\
&\le \int_{\mathbb{R}^3} |\nabla u|^2 dx + |\{V < b\}|^{\frac{2}{3}} \left(\int_{\{V<b\}} |u|^6 dx \right)^{\frac{1}{3}} + b^{-1} \int_{\{V \ge b\}} V(x)u^2 dx \\
&\le \left(1 + |\{V < b\}|^{\frac{2}{3}} S^{-1}\right) \int_{\mathbb{R}^3} |\nabla u|^2 dx + b^{-1} \int_{\mathbb{R}^3} V^+(x)u^2 dx \\
&\le \max\left\{1 + |\{V < b\}|^{\frac{2}{3}} S^{-1}, b^{-1}\right\} \int_{\mathbb{R}^3} \left(|\nabla u|^2 + V^+(x)u^2\right) dx,
\end{aligned}$$

which implies that the embedding $E \hookrightarrow H^1(\mathbb{R}^3)$ is continuous. Thus for each $s \in [2, 6]$, there exists $d_s > 0$ (independent of $\lambda \ge 1$) such that

$$|u|_s \le d_s\|u\| \le d_s\|u\|_\lambda, \quad \text{for } u \in E. \tag{2.1}$$

Let

$$F = \left\{u \in E : supp\, u \subset V^{-1}([0, \infty))\right\}$$

and denote the orthogonal complement of F in E_λ by $F_\lambda^\perp$. If $V \ge 0$, then $E = F$, otherwise $F_\lambda^\perp \ne \{0\}$. Let $A_\lambda = -\Delta + \lambda V$, then A_λ is formally self-adjoint in $L^2(\mathbb{R}^3)$ and the associated bilinear form

$$B_\lambda(u, v) = \int_{\mathbb{R}^3} (\nabla u \nabla v + \lambda V(x)uv)\, dx$$

is continuous in E_λ. For fixed $\lambda > 0$, consider the eigenvalue problem

$$-\Delta u + \lambda V^+(x)u = \mu\lambda V^-(x)u, \quad u \in F_\lambda^\perp. \tag{2.2}$$

Since $supp\, V^-$ is of finite measure, we deduce that the functional $\mathcal{T}(u) = \int_{\mathbb{R}^3} V^-(x)u^2 dx$ for $u \in F_\lambda^\perp$ is weakly continuous. Therefore, as a result of Theorems 4.45 and 4.46 of Ref. 38, we have the following proposition, which is the spectral theorem for compact self-adjoint operations together with the Courant-Fischer minimax characterization of eigenvalues.

Proposition 2.1. Suppose that $V(x) < 0$ for some x, (V_1) and (V_2) hold, then for any fixed $\lambda > 0$, problem (2.2) has a sequence of positive eigenvalues $\{\mu_j(\lambda)\}_{j=1}^\infty$, which may be characterized by

$$\mu_j(\lambda) = \inf_{\dim X \ge j, X \subset F_\lambda^\perp} \sup \left\{ \lambda^{-1}\|u\|_\lambda^2 : u \in X, \int_{\mathbb{R}^3} V^-(x)u^2 dx = 1 \right\}, \quad j = 1, 2, \ldots.$$

Moreover, $\mu_1(\lambda) \le \mu_2(\lambda) \le \cdots \le \mu_j(\lambda) \xrightarrow{j} \infty$, and the corresponding eigenfunctions $\{e_j(\lambda)\}_{j=1}^\infty$, which may be chosen so that $\langle e_i(\lambda), e_j(\lambda)\rangle_\lambda = \delta_{ij}$, are a basic for $F_\lambda^\perp$.

For $\{\mu_j(\lambda)\}$ defined above, we have also following properties, see Ref. 13, [Lemma 2.1] .

Proposition 2.2. Suppose that $V(x) < 0$ for some x, (V_1) and (V_2) hold, then for each fixed $j \in \mathbb{N}$, we have that

(i) $\mu_j(\lambda) \to 0$ *as* $\lambda \to +\infty$*;*
(ii) $\mu_j(\lambda)$ *is a non-increasing continuous function of* λ*.*

Remark 2.1. For $\mu_1(\lambda)$ defined by Proposition 2.1, it follows from Proposition 2.2 that there exists $\Lambda_0 > 0$ large such that $\mu_1(\lambda) \le 1$ for all $\lambda \ge \Lambda_0$. Set

$$E_\lambda^- = span\left\{e_j(\lambda) : \mu_j(\lambda) \le 1\right\} \quad and \quad E_\lambda^+ = span\left\{e_j(\lambda) : \mu_j(\lambda) > 1\right\}. \tag{2.3}$$

Then $E_\lambda = E_\lambda^- \oplus E_\lambda^+ \oplus F$ is an orthogonal decomposition, the quadratic form B_λ is negative semidefinite on E_λ^-, positive definite in $E_\lambda^+ \oplus F$ and it is easy to see that $B_\lambda(u,v) = 0$ if u, v are in different subspaces of the above decomposition of E_λ.

By (2.3), we know that $\dim E_\lambda^- \ge 1$ for any $\lambda \ge \Lambda_0$. In addition, we have that $\dim E_\lambda^- < \infty$ for any fixed $\lambda > 0$, since $\mu_j(\lambda) \to \infty$ as $j \to \infty$.

It is well known that problem $(\mathcal{SP})_\lambda$ can be reduced to a single equation with a nonlocal term. Actually, for any $u \in E_\lambda$, the linear functional $T_u : D^{1,2}(\mathbb{R}^3) \to \mathbb{R}$ defined by

$$T_u(v) = \int_{\mathbb{R}^3} K(x)u^2 v\, dx$$

is continuous in $D^{1,2}(\mathbb{R}^3)$. In fact, if $K \in L^2(\mathbb{R}^3)$, by the Hölder inequality and (2.1), we conclude that there is a constant $C > 0$ (independent of $\lambda \ge 1$) such that

$$|T_u(v)| \le |K|_2 |u|_6^2 |v|_6 \le C|K|_2 \|u\|_\lambda^2 \|v\|_D, \tag{2.4}$$

while for $K \in L^\infty(\mathbb{R}^3)$, we have

$$|T_u(v)| \le |K|_\infty |u|_{12/5}^2 |v|_6 \le C|K|_\infty \|u\|_\lambda^2 \|v\|_D. \tag{2.5}$$

Then by the Riesz representation theorem, there exists a unique $\phi_u \in D^{1,2}(\mathbb{R}^3)$ such that

$$-\Delta\phi_u = K(x)u^2.$$

Moreover, we can write an integral expression for ϕ_u in the form

$$\phi_u(x) = \int_{\mathbb{R}^3} \frac{K(y)u^2(y)}{4\pi|x-y|} dy.$$

By (2.4), it is easy to see that if $K \in L^2(\mathbb{R}^3)$,

$$\|\phi_u\|_D^2 = \int_{\mathbb{R}^3} K(x)\phi_u u^2 dx \le C|K|_2^2 \|u\|_\lambda^4. \tag{2.6}$$

Similarly, if $K \in L^\infty(\mathbb{R}^3)$, (2.5) implies that

$$\|\phi_u\|_D^2 = \int_{\mathbb{R}^3} K(x)\phi_u u^2 dx \le C|K|_\infty^2 \|u\|_\lambda^4. \tag{2.7}$$

Then, to seek a weak solution $(u, \phi) \in E_\lambda \times D^{1,2}(\mathbb{R}^3)$ for problem $(\mathcal{SP})_\lambda$, it is sufficient to find a weak solution of the first equation of $(\mathcal{SP})_\lambda$ with $\phi = \phi_u$. For this purpose, we define the functional $I_\lambda : E_\lambda \to \mathbb{R}$ by

$$I_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + \lambda V(x)u^2\right) dx + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_u u^2 dx - \int_{\mathbb{R}^3} a(x)F(u)dx,$$

where $F(t) = \int_0^t f(\tau)d\tau$. It can be proved that $I_\lambda \in C^1(E_\lambda, \mathbb{R})$ with

$$\left\langle I_\lambda'(u), v\right\rangle = \int_{\mathbb{R}^3} (\nabla u \nabla v + \lambda V(x)uv + K(x)\phi_u uv - a(x)f(u)v)\, dx, \quad \forall v \in E_\lambda.$$

We refer the readers to D'Aprile and Mugnai[10] for the detail.

Now, we are ready to give some properties of ϕ_u, see, e.g., Ref. 42, [Lemma 2.1]

Lemma 2.3. (i) *Let* $K \in L^\infty(\mathbb{R}^3)$*. If* $u_n \rightharpoonup u$ *in* $H^1(\mathbb{R}^3)$*, then*

$$\phi_{u_n} \rightharpoonup \phi_u \quad \text{in } D^{1,2}(\mathbb{R}^3).$$

(ii) *Let* $K \in L^2(\mathbb{R}^3)$*. If* $u_n \rightharpoonup u$ *in* $H^1(\mathbb{R}^3)$*, then up to a subsequence,*

$$\phi_{u_n} \to \phi_u \quad \text{in } D^{1,2}(\mathbb{R}^3).$$

In the sequel, we investigate the compactness conditions for the functional I_λ. Recall that a C^1 functional Φ satisfies Cerami condition at level c ($(C)_c$ condition for short) if any sequence $\{u_n\} \subset E$ such that $\Phi(u_n) \to c$ and $(1 + \|u_n\|)\Phi'(u_n) \to 0$ has a convergent subsequence, and such sequence is called a $(C)_c$ sequence. We begin with following Lemmas 2.4, 2.5, and 2.6, which ensure the boundedness of a $(C)_c$ sequence.

Lemma 2.4. Suppose that (V_1), (V_2), (F_1), (F_2), *and* (A_1) *hold. If* $K(x) > 0$ *for* $x \in \mathbb{R}^3$ *and* $K \in L^2(\mathbb{R}^3)$*, then every* $(C)_c$ *sequence of* I_λ *is bounded in* E_λ *for each* $c \in \mathbb{R}$ *and* $\lambda \geq 1$.

Proof. Let $\{u_n\} \subset E_\lambda$ be a $(C)_c$ sequence of I_λ, that is

$$I_\lambda(u_n) \to c, \quad (1 + \|u_n\|)\|I'_\lambda(u_n)\|_{E^*_\lambda} \to 0, \tag{2.8}$$

where E^*_λ denotes the dual space of E_λ. Arguing by contradiction, let $\|u_n\|_\lambda \to +\infty$ as $n \to \infty$. Set $\omega_n := u_n/\|u_n\|_\lambda$. Clearly, $\{\omega_n\}$ is bounded in E_λ and there exists $\omega \in H^1(\mathbb{R}^3)$ such that, up to a subsequence,

$$\begin{cases} \omega_n \rightharpoonup \omega & \text{in } E_\lambda, \\ \omega_n \to \omega & \text{in } L^s_{loc}(\mathbb{R}^3), \quad \text{for all } s \in [1,6), \\ \omega_n \to \omega & \text{a.e. in } \mathbb{R}^3. \end{cases} \tag{2.9}$$

Case 1: $\omega \equiv 0$ in E_λ. On one hand, since $\|u_n\|_\lambda \to +\infty$ as $n \to \infty$, it follows from (2.8) that

$$\frac{\langle I'_\lambda(u_n), u_n\rangle}{\|u_n\|^2_\lambda} = o(1),$$

that is,

$$\begin{aligned} o(1) &= \|\omega_n\|^2_\lambda - \int_{\mathbb{R}^3} \lambda V^-(x)\omega_n^2 dx + \int_{\mathbb{R}^3} K(x)\phi_{\omega_n} u_n^2 dx - \int_{\mathbb{R}^3} a(x)\frac{f(u_n)}{u_n}\omega_n^2 dx \\ &\geq 1 - \int_{\mathbb{R}^3} a(x)\frac{f(u_n)}{u_n}\omega_n^2 dx + o(1), \end{aligned}$$

where, and in what follows, $o(1)$ denotes a quantity which goes the zero as $n \to \infty$. Hence, we yield that

$$\limsup_{n\to\infty} \int_{\mathbb{R}^3} a(x)\frac{f(u_n)}{u_n}\omega_n^2 dx \geq 1. \tag{2.10}$$

On the other hand, from (A_1) we see that there exists $\theta \in (0,1)$ such that

$$\sup\{f(s)/s : s > 0\} \leq \theta \inf\{V(x)/a(x) : |x| > R_0 \text{ and } V(x) \geq b\}. \tag{2.11}$$

It then follows from (V_2) that there exists $R_1 > R_0$ such that for any $n \in \mathbb{N}$,

$$\int_{B^c_{R_1}(0)\cap\{V<b\}} a(x)\frac{f(u_n)}{u_n}\omega_n^2 dx < 1 - \theta. \tag{2.12}$$

Thus, by (2.11) and (2.12), we yield that for any $n \in \mathbb{N}$,

$$\int_{|x|\geq R_1} a(x)\frac{f(u_n)}{u_n}\omega_n^2 dx < \theta \int_{B^c_{R_1}(0)\cap\{V\geq b\}} \lambda V^+(x)\omega_n^2 dx + (1-\theta) \leq 1. \tag{2.13}$$

By (2.9), we see that $\omega_n \to 0$ in $L^2\left(B_{R_1}(0)\right)$. Then, it follows from (F_1), (F_2) and (A_1) that

$$\int_{|x|\le R_1} a(x)\frac{f(u_n)}{u_n}\omega_n^2 dx \le C_1|a|_\infty \int_{|x|\le R_1} \omega_n^2 dx \overset{n}{\to} 0. \tag{2.14}$$

Combining (2.13) with (2.14), we deduce that

$$\limsup_{n\to\infty} \int_{\mathbb{R}^3} a(x)\frac{f(u_n)}{u_n}\omega_n^2 dx < 1,$$

which contradicts (2.10). So, $\omega \not\equiv 0$.

Case 2: $\omega \not\equiv 0$ in E_λ. Since $\|u_n\|_\lambda \to +\infty$ as $n \to \infty$, it follows from (2.8) that

$$\frac{\langle I'_\lambda(u_n), u_n\rangle}{\|u_n\|_\lambda^4} = o(1),$$

that is,

$$o(1) = \frac{1}{\|u_n\|_\lambda^2} + \int_{\mathbb{R}^3} K(x)\phi_{\omega_n}\omega_n^2 dx - \frac{\int_{\mathbb{R}^3} \frac{f(x,u_n)}{u_n}\omega_n^2 dx}{\|u_n\|_\lambda^2} + o(1).$$

This together with (F_1) and (F_2), we obtain

$$\int_{\mathbb{R}^3} K(x)\phi_{\omega_n}\omega_n^2 dx = o(1). \tag{2.15}$$

By Lemma 2.3, we have

$$\int_{\mathbb{R}^3} K(x)\phi_{\omega_n}\omega_n^2 dx = \int_{\mathbb{R}^3} K(x)\phi_\omega \omega^2 dx + o(1). \tag{2.16}$$

Hence, (2.15) and (2.16) imply that

$$\int_{\mathbb{R}^3} K(x)\phi_\omega \omega^2 dx = o(1).$$

Therefore, $\omega \equiv 0$, since $K(x) > 0$ for $x \in \mathbb{R}^3$. This is a contradiction.

So, $\{u_n\}$ is bounded in E_λ. This completes the proof of the lemma. □

Lemma 2.5. Suppose that (V_1), (V_2), (F_1), (F_2), and (A_1) hold. If $K(x) \ge k_0 > 0$ for $x \in \mathbb{R}^3$ and $K \in L^\infty(\mathbb{R}^3)$, then every $(C)_c$ sequence of I_λ is bounded in E_λ for each $c \in \mathbb{R}$ and $\lambda \ge 1$.

Proof. For any fixed $R > 0$, let $\xi_R : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function such that $0 \le \xi_R \le 1$ and

$$\xi_R(x) = \begin{cases} 0, & 0 \le |x| \le R, \\ 1, & |x| \ge 2R, \end{cases}$$

Moreover, there exists a constant $C_0 > 0$ (independent of R) such that

$$|\nabla \xi_R(x)| \le \frac{C_0}{R}, \quad \text{for all } x \in \mathbb{R}^3. \tag{2.17}$$

Then for some constant $C_1 > 0$ (independent of R), we have

$$\|\xi_R u\|_\lambda \le C_1\|u\|_\lambda, \quad \text{for all } u \in E_\lambda \text{ and } R \ge 1. \tag{2.18}$$

Let $\{u_n\} \subset E_\lambda$ be a $(C)_c$ sequence of I_λ. By (2.8) and (2.18), we obtain

$$\left\langle I'_\lambda(u_n), \xi_R u_n \right\rangle = o(1), \tag{2.19}$$

i.e.,

$$\int_{\mathbb{R}^3} (|\nabla u_n|^2 + \lambda V(x)u_n^2)\xi_R dx + \int_{\mathbb{R}^3} u_n \nabla u_n \nabla \xi_R dx + \int_{\mathbb{R}^3} K(x)\phi_{u_n}u_n^2\xi_R dx$$
$$= \int_{\mathbb{R}^3} a(x)f(u_n)u_n\xi_R dx + o(1). \tag{2.20}$$

From (2.11), we deduce that for all $n \in \mathbb{N}$ and $R \geq R_0$,

$$\int_{\mathbb{R}^3} a(x) f(u_n) u_n \xi_R dx \leq \theta \int_{\{V \geq b\}} \lambda V^+(x) u_n^2 \xi_R dx + C_1 \int_{\{V < b\}} u_n^2 \xi_R dx.$$

Moreover, by (2.17) one has that for all $n \in \mathbb{N}$ and $R \geq R_0$,

$$\begin{aligned}\int_{\mathbb{R}^3} |u_n \nabla u_n \nabla \xi_R| \, dx &\leq \frac{C_0}{R} \left(\int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \int_{\mathbb{R}^3} |u_n|^2 dx \right) \\ &\leq \frac{C_2}{R} \|u_n\|_\lambda^2.\end{aligned}$$

Then, (2.20) becomes that for all $R \geq R_0$,

$$\begin{aligned}\int_{\mathbb{R}^3} (|\nabla u_n|^2 + (1-\theta)\lambda V^+(x) u_n^2) \xi_R dx &\leq \frac{C_2}{R} \|u_n\|_\lambda^2 + C_\lambda \int_{\{V < b\}} u_n^2 \xi_R dx + o(1) \\ &\leq \left(\frac{C_2}{R} + C_\lambda \left| B_R^c(0) \cap \{V < b\} \right|^{2/3} \right) \|u_n\|_\lambda^2 + o(1). \qquad (2.21)\end{aligned}$$

Similar to (2.19), we see that $\langle I'_\lambda(u_n), u_n \rangle = o(1)$, i.e.,

$$\int_{\mathbb{R}^3} \left(|\nabla u_n|^2 + \lambda V(x) u_n^2 + K(x) \phi_{u_n} u_n^2 - a(x) f(u_n) u_n \right) dx = o(1). \qquad (2.22)$$

Since $-\Delta \phi_{u_n} = K(x) u_n^2$, multiplying this equation by $|u_n|$, integrating by parts and using the Young's inequality, we have

$$\begin{aligned}k_0 \int_{\mathbb{R}^3} |u_n|^3 dx \leq \int_{\mathbb{R}^3} K(x) |u_n|^3 dx &= \int_{\mathbb{R}^3} \nabla \phi_{u_n} \nabla u_n dx \\ &\leq \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u_n|^2 + K(x) \phi_{u_n} u_n^2) dx. \qquad (2.23)\end{aligned}$$

It then follows from (2.22) and (2.23) that

$$\frac{1}{2} \int_{\mathbb{R}^3} \left(|\nabla u_n|^2 + \lambda V^+(x) u_n^2 \right) dx + \int_{\mathbb{R}^3} h_\lambda(u_n) dx \leq o(1),$$

i.e.,

$$\frac{1}{2} \|u_n\|_\lambda^2 \leq o(1) - \int_{\mathbb{R}^3} h_\lambda(u_n) dx, \qquad (2.24)$$

where $h_\lambda(u_n) = k_0 |u_n|^3 - \delta_\lambda u_n^2$. By (2.21), for some constant $C_3 > 0$ (independent of R), we have

$$\int_{|x| \geq R} \lambda V^+(x) u_n^2 dx \leq \left(\frac{C_3}{R} + C_\lambda \left| B_R^c(0) \cap \{V < b\} \right|^{2/3} \right) \|u_n\|_\lambda^2 + o(1). \qquad (2.25)$$

Let $\beta_\lambda = \inf_{t \in \mathbb{R}} h_\lambda(t)$. Then $\beta_\lambda \in (-\infty, 0)$ and (2.25) yields

$$\begin{aligned}\int_{\mathbb{R}^3} h_\lambda(u_n) dx &\geq \int_{|x| \leq R} \beta_\lambda dx - \delta_\lambda \int_{|x| \geq R} u_n^2 dx \\ &\geq \beta_\lambda |B_R(0)| - \delta_\lambda b^{-1} \int_{|x| \geq R} \lambda V^+(x) u_n^2 dx - C_\lambda \left| B_R^c(0) \cap \{V(x) < b\} \right|^{2/3} \|u_n\|_\lambda^2 \\ &\geq \beta_\lambda |B_R(0)| - C_\lambda \left(\frac{1}{R} + \left| B_R^c(0) \cap \{V(x) < b\} \right|^{2/3} \right) \|u_n\|_\lambda^2 + o(1). \qquad (2.26)\end{aligned}$$

Using (2.24) and (2.26), we see that

$$\frac{1}{2} \|u_n\|_\lambda^2 \leq |\beta_\lambda| \, |B_R(0)| + C_\lambda \left(\frac{1}{R} + \left| B_R^c(0) \cap \{V(x) < b\} \right|^{2/3} \right) \|u_n\|_\lambda^2 + o(1). \qquad (2.27)$$

Note that the constant C_λ is independent of R, we can choose $R \geq R_0$ large enough such that $C_\lambda \left(\frac{1}{R} + \left| B_R^c(0) \cap \{V(x) < b\} \right|^{2/3} \right) < \frac{1}{2}$. Then, (2.27) implies that $\{u_n\}$ is bounded in E_λ. This completes the proof of the lemma. $\square$

Lemma 2.6. Suppose that $V \geq 0$, (V_1), (V_2), (F_4) and (A_2) hold. If $K(x) \geq 0$ for $x \in \mathbb{R}^3$ and $K \in L^2(\mathbb{R}^3) \cup L^\infty(\mathbb{R}^3)$, then every $(C)_c$ sequence of I_λ is bounded in E_λ for each $c > 0$ and $\lambda > 0$.

Proof. For n large enough, by (A_2) and (F_4), we have

$$\begin{aligned} c + 1 &\geq I_\lambda(u_n) - \frac{1}{4}\langle I'_\lambda(u_n), u_n\rangle \\ &= \frac{1}{4}\|u\|_\lambda^2 + \frac{1}{4}\int_{\mathbb{R}^3} a(x)\,[f(u_n)u_n - 4F(u_n)]\,dx \\ &\geq \frac{1}{4}\|u\|_\lambda^2. \end{aligned}$$

Therefore, $\{u_n\}$ is bounded in E_λ. This completes the proof of the lemma. □

Now, we are ready to give our first compactness result, which is very useful in the proof of Theorems 1.1 and 1.2.

Lemma 2.7. Suppose that (V_1), (V_2), (F_1), (F_2), and (A_1) hold. If $K(x) > 0$ for $x \in \mathbb{R}^3$ and $K \in L^2(\mathbb{R}^3)$ ($K(x) \geq k_0 > 0$ for $x \in \mathbb{R}^3$ and $K \in L^\infty(\mathbb{R}^3)$), then I_λ satisfies $(C)_c$ condition for all $c \in \mathbb{R}$ and $\lambda \geq 1$.

Proof. By Lemma 2.4 (or Lemma 2.5), we see that the $(C)_c$ sequence $\{u_n\}$ given by (2.8) is bounded in E_λ. It then follows from (2.21) that for any $\epsilon > 0$, there exist $R(\epsilon) > R_0$ and $n(\epsilon) > 0$ such that

$$\int_{|x|\geq R} \left(|\nabla u_n|^2 + \lambda V^+(x)u_n^2\right)dx \leq \epsilon, \tag{2.28}$$

for all $R \geq R(\epsilon)$ and $n \geq n(\epsilon)$. Since $\{u_n\}$ is bounded in E_λ, passing to a subsequence if necessary, we may assume that for some $u \in E_\lambda$,

$$\begin{cases} u_n \rightharpoonup u & \text{in } E_\lambda, \\ u_n \to u & \text{in } L^s_{loc}(\mathbb{R}^3), \text{ for all } s \in [1,6). \end{cases} \tag{2.29}$$

Therefore, in order to prove this lemma, it is sufficient to show that $\|u_n\|_\lambda \to \|u\|_\lambda$ as $n \to \infty$. It follows from $\langle I'_\lambda(u_n), u\rangle = o(1)$ and $\langle I'_\lambda(u_n), u_n\rangle = o(1)$ that

$$\|u\|_\lambda^2 - \int_{\mathbb{R}^3} \lambda V^-(x)u_n u dx + \int_{\mathbb{R}^3} K(x)\phi_{u_n} u_n u dx - \int_{\mathbb{R}^3} a(x)f(u_n)u dx = o(1) \tag{2.30}$$

and

$$\|u_n\|_\lambda^2 - \int_{\mathbb{R}^3} \lambda V^-(x)u_n^2 dx + \int_{\mathbb{R}^3} K(x)\phi_{u_n} u_n^2 dx - \int_{\mathbb{R}^3} a(x)f(u_n)u_n dx = o(1). \tag{2.31}$$

We claim that

$$\left.\begin{aligned} \int_{\mathbb{R}^3} \lambda V^-(x)u_n(u_n - u)dx = o(1), \\ \int_{\mathbb{R}^3} a(x)f(u_n)(u_n - u)dx = o(1), \\ \int_{\mathbb{R}^3} K(x)\phi_{u_n} u_n(u_n - u)dx = o(1). \end{aligned}\right\} \tag{2.32}$$

In fact, since $\|u_n\|_\lambda$ is bounded in E_λ, by the Hölder inequality and (2.29), we deduce that

$$\left|\int_{\mathbb{R}^3} \lambda V^-(x)u_n(u_n - u)dx\right| \leq \lambda|V^-|_\infty \int_{\{V<b\}} |u_n||u_n - u|dx \xrightarrow{n} 0.$$

Since $\|u_n\|_\lambda$ is bounded in E_λ, by (A_1), the Hölder inequality and noting (2.28) and (2.29), we have

$$\begin{aligned}\left|\int_{\mathbb{R}^3} a(x)f(u_n)(u_n-u)dx\right| &\le \left|\int_{|x|\le R} a(x)f(u_n)(u_n-u)dx\right| + \left|\int_{|x|\ge R} a(x)f(u_n)(u_n-u)dx\right|\\ &\le C|u_n-u|_{L^2(B_R(0))} + C\int_{\{V<b\}}|u_n-u|^2 + C\int_{|x|\ge R}\lambda V^+(x)u_n^2dx\\ &\to 0, \quad \text{by letting } n\to\infty \text{ and then } R\to\infty.\end{aligned}$$

On the other hand, if $K\in L^\infty(\mathbb{R}^3)$, by the Hölder inequality, and noting (2.28) and (2.29), we yield

$$\begin{aligned}\left|\int_{\mathbb{R}^3} K(x)\phi_{u_n}u_n(u_n-u)dx\right| &\le \left|\int_{|x|\le R} K(x)\phi_{u_n}u_n(u_n-u)dx\right| + \left|\int_{|x|\ge R} K(x)\phi_{u_n}u_n(u_n-u)dx\right|\\ &\le |K|_\infty|\phi_{u_n}|_{L^6(B_R(0))}|u_n|_{L^3(B_R(0))}|u_n-u|_{L^2(B_R(0))}\\ &\quad + |K|_\infty|\phi_{u_n}|_{L^6(|x|\ge R)}|u_n|_{L^2(|x|\ge R)}|u_n-u|_{L^3(|x|\ge R)}\\ &\le C|u_n-u|_{L^2(B_R(0))} + C|u_n|_{L^2(|x|\ge R)}\\ &\to 0, \quad \text{by letting } n\to\infty \text{ and then } R\to\infty,\end{aligned}$$

while for $K\in L^2(\mathbb{R}^3)$, we obtain

$$\begin{aligned}\left|\int_{\mathbb{R}^3} K(x)\phi_{u_n}u_n(u_n-u)dx\right| &\le |u_n|_6|\phi_{u_n}|_6\left(\int_{\mathbb{R}^3}|K(x)(u_n-u)|^{3/2}dx\right)^{2/3}\\ &\le C\left(\int_{\mathbb{R}^3}|K(x)(u_n-u)|^{3/2}dx\right)^{2/3} \xrightarrow{n} 0,\end{aligned} \tag{2.33}$$

since $|u_n-u|^{3/2}\xrightarrow{n} 0$ in $L^4(\mathbb{R}^3)$ and $K(x)^{3/2}\in L^{4/3}(\mathbb{R}^3)$. So, (2.32) is proved. By (2.32), it is easy to see that (2.30) and (2.31) imply that $\|u_n\|_\lambda \xrightarrow{n} \|u\|_\lambda$. Thus, $u_n\to u$ in E_λ. This completes the proof of the lemma. □

We remark that (1.3) is important in the proof of Lemma 2.7. Since the condition (A_2) does not contain (1.3), we have to find another compactness result to prove the Theorem 1.4. In fact, we have the following parameter dependent compactness result.

Lemma 2.8. Suppose that $V\ge 0$, (V_1), (V_2), (F_1), (F_3), (F_4), and (A_2) hold. If $K(x)\ge 0$ for $x\in\mathbb{R}^3$, $K\in L^2(\mathbb{R}^3)\cup L^\infty(\mathbb{R}^3)$, then for any $M>0$, there exists $\overline{\Lambda}=\overline{\Lambda}(M)>0$ such that I_λ satisfies $(C)_c$ condition for all $c\in(0,M)$ and $\lambda\ge\overline{\Lambda}$.

Proof. We give the proof for the case $K\in L^2(\mathbb{R}^3)$, and the proof is similar for the case $K\in L^\infty(\mathbb{R}^3)$. Let $\{u_n\}$ be a $(C)_c$ sequence with $c\in(0,M)$. By Lemma 2.6, we know that $\{u_n\}$ is bounded in E_λ. Therefore, up to a subsequence, we may assume that for some $u\in E_\lambda$,

$$\begin{cases} u_n \rightharpoonup u \text{ in } E_\lambda,\\ u_n\to u \text{ in } L^s_{loc}(\mathbb{R}^3) \text{ for all } s\in[1,6).\end{cases}$$

First, it is easy to check that $I'_\lambda(u)=0$. Moreover, by (A_2) and (F_4), we have

$$\begin{aligned} I_\lambda(u) &= \frac{1}{4}\|u\|_\lambda^2 + \frac{1}{4}\int_{\mathbb{R}^3} a(x)\,[f(u)u-4F(u)]\,dx\\ &\ge \frac{1}{4}\|u\|_\lambda^2 \ge 0.\end{aligned} \tag{2.34}$$

Now, we show that $u_n\to u$ in E_λ. Let $v_n := u_n - u$. By (V_2) , we have

$$|v_n|_2^2 = \int_{\{V\ge b\}} v_n^2dx + \int_{\{V<b\}} v_n^2dx \le \frac{1}{\lambda b}\|v_n\|_\lambda^2 + o(1). \tag{2.35}$$

It then follows from the Hölder and Sobolev inequalities that

$$|v_n|_p \le |v_n|_2^\theta|v_n|_6^{1-\theta} \le d|v_n|_2^\theta|\nabla v_n|_2^{1-\theta} \le d(\lambda b)^{-\frac{\theta}{2}}\|v_n\|_\lambda + o(1), \tag{2.36}$$

where $p \in (2,6)$, $\theta = \frac{6-p}{2p}$ and the constant d is independent of λ. By Brezis-Lieb Lemma, we obtain

$$I_\lambda(v_n) = I_\lambda(u_n) - I_\lambda(u) + o(1) \quad \text{and} \quad I'_\lambda(v_n) = I'_\lambda(u_n) + o(1). \tag{2.37}$$

Consequently, this together with (2.34), we obtain

$$\begin{aligned} \frac{1}{4}\|v_n\|_\lambda^2 &\le I_\lambda(v_n) - \frac{1}{4}\langle I'_\lambda(v_n), v_n\rangle \\ &= c - I_\lambda(u) + o(1) \\ &\le M + o(1). \end{aligned} \tag{2.38}$$

From (2.1) and (2.38), we yield

$$|v_n|_p^2 \le d_p^2\|v_n\|_\lambda^2 \le 4d_p^2 M + o(1), \tag{2.39}$$

where the constant $d_p > 0$ is independent of $\lambda \ge 1$. Since $\langle I'_\lambda(v_n), v_n\rangle = o(1)$ and

$$\int_{\mathbb{R}^3} a(x) f(v_n) v_n dx \le \epsilon\|v_n\|_\lambda^2 + C_\epsilon |v_n|_4^4,$$

it follows from (2.36) and (2.39) that

$$\begin{aligned} o(1) &= \|v_n\|_\lambda^2 + \int_{\mathbb{R}^3} K(x)\phi_{v_n} v_n^2 dx - \int_{\mathbb{R}^3} a(x) f(v_n) v_n \\ &\ge \frac{1}{2}\|v_n\|_\lambda^2 - C|v_n|_4^2 |v_n|_4^2 \\ &\ge \left(\frac{1}{2} - 4Cd_4^2 M d^2(\lambda b)^{-\theta}\right)\|v_n\|_\lambda^2 + o(1). \end{aligned}$$

Thus, there exists $\overline{\Lambda} = \overline{\Lambda}(M) > 0$ such that $v_n \to 0$ in E_λ as $n \to \infty$, for $\lambda \ge \overline{\Lambda}$. This completes the proof of the lemma. □

III. PROOF OF THEOREMS 1.1-1.4

In this section, we study the existence of solutions for system $(\mathcal{SP})_\lambda$ and give the proof of Theorems 1.1–1.4. If V is sign-changing, we show that the functional I_λ have the linking geometry to apply the following linking theorem.[26]

Proposition 3.1. Let $E = E_1 \oplus E_2$ *be a Banach space with* $\dim E_2 < \infty$, $\Phi \in C^1(E, \mathbb{R})$. *If there exist* $R > \rho > 0$, $\kappa > 0$ *and* $e_0 \in E_1$ *such that*

$$\kappa = \inf \Phi(E_1 \cap S_\rho) > \sup \Phi(\partial Q),$$

where $S_\rho = \{u \in E : \|u\| = \rho\}$, $Q = \{u = v + te_0 : v \in E_2,\ t \ge 0,\ \|u\| \le R\}$. *Then* Φ *has a* $(C)_c$ *sequence with* $c \in [\kappa, \sup \Phi(Q)]$.

In the sequel, we use Proposition 3.1 with $E_1 = E_\lambda^+ \oplus F$ and $E_2 = E_\lambda^-$. By Remark 2.1, there exists $\Lambda_0 > 0$ such that $E_\lambda^- \ne \emptyset$ and is finite dimensional for $\lambda \ge \Lambda_0$. Now we are ready to show the linking structure of the functional I_λ.

Lemma 3.2. Suppose that (V_1), (V_2), (F_1), (F_2), *and* (A_2) *are satisfied. If* $K(x) \ge 0$ *for* $x \in \mathbb{R}^3$ *and* $K \in L^2(\mathbb{R}^3) \cup L^\infty(\mathbb{R}^3)$, *then for each* $\lambda > \Lambda_0$, *there exist* $\rho_\lambda > 0$ *and* $\kappa_\lambda > 0$ *such that*

$$I_\lambda(u) \ge \kappa_\lambda, \quad \forall u \in E_\lambda^+ \oplus F \text{ with } \|u\|_\lambda = \rho_\lambda.$$

Moreover, if $V \ge 0$, *we can choose* ρ *and* κ *independent of* $\lambda \ge 1$.

Proof. From (F_1) and (F_2), we deduce that for any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$|F(s)| \le \epsilon|s|^2 + C_\epsilon|s|^{2^*}. \tag{3.1}$$

Observe that, from the definition of E_λ^+, there exists $\delta_\lambda > 0$ such that

$$B_\lambda(u,u) \ge \delta_\lambda\|u\|_\lambda^2, \quad \text{for } u \in E_\lambda^+$$

and

$$B_\lambda(u,u) = \|u\|_\lambda^2, \quad \text{for } u \in F.$$

Thus, for $u = v + w \in E_\lambda^+ \oplus F$, since $(v,w)_\lambda = 0$, it follows from (3.1) and (2.1) that

$$\begin{aligned} I_\lambda(u) &= \frac{1}{2}B_\lambda(v,v) + \frac{1}{2}B_\lambda(w,w) + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_u u^2 dx - \int_{\mathbb{R}^3} a(x)F(u)dx \\ &\geq \frac{1}{2}\min\{\delta_\lambda, 1\}\|u\|_\lambda^2 - \int_{\mathbb{R}^3} a(x)F(u)dx \\ &\geq \frac{1}{2}\min\{\delta_\lambda, 1\}\|u\|_\lambda^2 - \epsilon|a|_\infty d_2^2\|u\|_\lambda^2 - C_\epsilon|a|_\infty d_6^2\|u\|_\lambda^{2^*}. \end{aligned}$$

Taking $\epsilon > 0$ small enough, we have the desired conclusion. If $V \geq 0$, since $B_\lambda(u,u) = \|u\|_\lambda^2$, we can choose $\rho > 0$ and $\kappa > 0$ independent of $\lambda \geq 1$. The proof is complete. □

Lemma 3.3. Suppose that $V(x) < 0$ for some x, (V_1), (V_2), (F_1), (F_2), and (A_2) are satisfied. If $K(x) \geq 0$ for $x \in \mathbb{R}^3$ and $K \in L^2(\mathbb{R}^3) \cup L^\infty(\mathbb{R}^3)$, then for each $m \in \mathbb{N}$, there exist $\lambda_m > m$, $k_{\lambda_m}^* > 0$, $w_m \in E_{\lambda_m}^+ \oplus F$ and $R_{\lambda_m} > \rho_{\lambda_m}$ such that for $|K|_2 < k_{\lambda_m}^*$ $\left(\text{or } |K|_\infty < k_{\lambda_m}^*\right)$

$$\sup_{u\in\partial Q} I_{\lambda_m}(u) < \kappa_{\lambda_m},$$

where $Q = \left\{u = v + tw_m : v \in E_{\lambda_m}^-, t \geq 0, \|u\|_{\lambda_m} \leq R_{\lambda_m}\right\}$.

Proof. We give the proof for $K \in L^2(\mathbb{R}^3)$, the proof for $K \in L^\infty(\mathbb{R}^3)$ is similar.

(i) For each $m \in \mathbb{N}$, we may find $j_m \in \mathbb{N}$ such that $\mu_{j_m}(m) > 1$, since $\mu_j(m) \xrightarrow{j} \infty$. Set

$$\Omega_0 = \left\{x \in \mathbb{R}^3 : V(x) < 0\right\}.$$

Since $V(x) < 0$ for some x, we see that $\Omega_0 \neq \emptyset$. We claim that there exists $\lambda_m > m$ such that

$$1 < \mu_{j_m}(\lambda_m) < 1 + \frac{1}{2}a_0 l\left(\lambda_m|V^-|_\infty\right)^{-1}, \tag{3.2}$$

where $l > 0$ is given by (F_2). Indeed, for the above fixed j_m, by Proposition 2.2, there exists $\Lambda_m > m$ such that $\mu_{j_m}(\Lambda_m) = 1$. Then by the continuity we see that there exists $\lambda_m \in (m, \Lambda_m)$ such that $\mu_{j_m}(\lambda_m) = 1 + \delta_m$, where $0 < \delta_m < \min\left\{\mu_{j_m}(m) - 1, \frac{a_0 l}{2\Lambda_m|V^-|_\infty}\right\}$. So, (3.2) holds. Let $e_{j_m}(\lambda_m)$ be an eigenfunction of $\mu_{j_m}(\lambda_m)$ and we denote $e_{j_m}(\lambda_m)$ by w_m for simplicity. Since $\mu_{j_m}(\lambda_m) > 1$, we have $w_m \in E_{\lambda_m}^+$. Let

$$G(x,u) = a(x)F(u) - \frac{l}{2}a(x)u^2 \quad \text{and} \quad |u|_{2,a}^2 = \int_{\mathbb{R}^3} a(x)u^2 dx.$$

Moreover, we set

$$J_\lambda(u) = \frac{1}{2}B_\lambda(u,u) - \int_{\mathbb{R}^3} a(x)F(u)dx, \quad \text{for } u \in E_\lambda.$$

Then for $u \in E_{\lambda_m}^- \oplus \mathbb{R}w_m$, we have

$$J_{\lambda_m}(u) = \frac{1}{2}\int_{\mathbb{R}^3}\left[|\nabla u|^2 + (\lambda_m V(x) - la(x))\, u^2\right] dx - \int_{\mathbb{R}^3} G(x,u)dx.$$

From (F_1) and (F_2), we conclude that for $v \in E_{\lambda_m}^- \oplus \mathbb{R}w_m$ with $|v|_{2,a} = 1$, there holds

$$\lim_{t\to+\infty}\frac{1}{t^2}\int_{\mathbb{R}^3} G(x,tv)dx = 0.$$

Thus for any $\epsilon > 0$, there exists $R(\epsilon) > 0$ such that

$$\left|\int_{\mathbb{R}^3} G(x,u)dx\right| \leq \epsilon\int_{\mathbb{R}^3} a(x)u^2 dx,$$

for all $u \in E^-_{\lambda_m} \oplus \mathbb{R}w_m$ with $|u|_{2,a} \geq R(\epsilon)$. Hence, for $l > 0$ given by (F_2), there exists $R(l) > 0$ such that for all $u \in E^-_{\lambda_m} \oplus \mathbb{R}w_m$ with $|u|_{2,a} \geq R(l)$, there holds

$$\begin{aligned} J_{\lambda_m}(u) &\leq \frac{1}{2}\left[\left(\mu_{j_m}(\lambda_m)-1\right)\int_{\mathbb{R}^3}\lambda_m V^-(x)u^2dx - l\int_{\mathbb{R}^3}a(x)u^2dx\right] + \left|\int_{\mathbb{R}^3}G(x,u)dx\right| \\ &\leq \frac{1}{2}\left[\left(\mu_{j_m}(\lambda_m)-1\right)\int_{\mathbb{R}^3}\lambda_m V^-(x)u^2dx - \frac{l}{2}\int_{\mathbb{R}^3}a(x)u^2dx\right] - \frac{l}{8}\int_{\mathbb{R}^3}a(x)u^2dx \\ &\leq \frac{1}{2}\left[\lambda_m|V^-|_\infty\left(\mu_{j_m}(\lambda_m)-1\right) - \frac{l}{2}a_0\right]\int_{\Omega_0}u^2dx - \frac{l}{8}\int_{\mathbb{R}^3}a(x)u^2dx \\ &\leq -\frac{l}{8}\int_{\mathbb{R}^3}a(x)u^2dx \quad \text{by (3.2).} \end{aligned} \tag{3.3}$$

Since $E^-_{\lambda_m} \oplus \mathbb{R}w_m$ is finite dimensional, we have $|u|_{2,a} \geq C_{\lambda_m}\|u\|_{\lambda_m}$ for all $u \in E^-_{\lambda_m} \oplus \mathbb{R}w_m$. Then, by (3.3), we see that there exists $R_{\lambda_m} > \rho_{\lambda_m}$ such that

$$J_{\lambda_m}(u) < 0, \qquad \text{for all } u \in E^-_{\lambda_m} \oplus \mathbb{R}w_m \text{ with } \|u\|_{\lambda_m} = R_{\lambda_m}.$$

Combining this with (2.6), we yield that for all $u \in E^-_{\lambda_m} \oplus \mathbb{R}w_m$ with $\|u\|_{\lambda_m} = R_{\lambda_m}$,

$$I_{\lambda_m}(u) = J_{\lambda_m}(u) + \frac{1}{4}\int_{\mathbb{R}^3}K(x)\phi_u u^2dx \leq C|K|_2^2\|u\|^4_{\lambda_m} = C|K|_2^2R^4_{\lambda_m}.$$

Consequently, choosing $k^*_{\lambda_m} = \kappa^{1/2}_{\lambda_m}/\left(C^{1/2}R^2_{\lambda_m}\right)$, then for $|K|_2 < k^*_{\lambda_m}$, we deduce that $I_{\lambda_m}(u) < \kappa_{\lambda_m}$ for all $u \in E^-_{\lambda_m} \oplus \mathbb{R}w_m$ with $\|u\|_{\lambda_m} = R_{\lambda_m}$.

(ii) It follows from the definition of $E^-_{\lambda_m}$ that, for $u \in E^-_{\lambda_m} \oplus \mathbb{R}w_m$ with $\|u\|_{\lambda_m} \leq R_{\lambda_m}$,

$$I_{\lambda_m}(u) \leq \frac{1}{4}\int_{\mathbb{R}^3}K(x)\phi_u u^2dx \leq C|K|_2^2\|u\|^4_{\lambda_m} \leq C|K|_2^2R^4_{\lambda_m}.$$

Therefore, for $|K|_2 < k^*_{\lambda_m}$, we conclude that $I_{\lambda_m}(u) < \kappa_{\lambda_m}$ for all $u \in E^-_{\lambda_m}$ with $\|u\|_{\lambda_m} \leq R_{\lambda_m}$. The proof is complete. □

Lemma 3.4. Suppose that $V \geq 0$, (V_1), (V_2), (F_1), (F_2) (or (F_3)), and (A_2) hold. If $K(x) \geq 0$ for $x \in \mathbb{R}^3$ and $K \in L^2(\mathbb{R}^3) \cup L^\infty(\mathbb{R}^3)$, then any nontrivial solution of $(\mathcal{SP})_\lambda$ is a positive solution.

Proof. Suppose u is a nontrivial solution of problem $(\mathcal{SP})_\lambda$, then

$$\int_{\mathbb{R}^3}\left(\nabla u\nabla v + \lambda V(x)uv + K(x)\phi_u uv\right)dx = \int_{\mathbb{R}^3}a(x)f(u)v\,dx, \quad \forall v \in E_\lambda.$$

Taking $v = u^- = -\min\{0,u\}$, and by (F_1) we obtain

$$\int_{\mathbb{R}^3}\left[\left(|\nabla u^-|^2 + \lambda V(x)|u^-|^2\right) + K(x)\phi_u|u^-|^2\right]dx = 0.$$

So, we see that $\|u^-\|_\lambda = 0$ and $u \equiv u^+ \geq 0$. By the strong maximum principle, we have $u > 0$. The proof is complete. □

Proof of Theorem 1.1. It is a direct consequence of Proposition 3.1 and Lemmas 3.2, 3.3, 2.4, 2.5, and 2.7. The proof is complete. □

Proof of Theorem 1.2. Since we suppose $V \geq 0$, the function I_λ has mountain pass geometry and the existence of nontrivial solutions can be obtained by the mountain pass theorem.[39] In fact, by Lemma 3.2 with $E^-_\lambda = \{0\}$, $0 \in E_\lambda$ is the local minimum for I_λ, and ρ, κ are independent of λ. Since $l > \mu^*$, we can choose a nonnegative function $\varphi \in H^1_0(\Omega)$ with

$$\int_\Omega a(x)\varphi^2dx = 1 \quad \text{and} \quad \int_\Omega|\nabla\varphi|^2dx < l.$$

It then follows from (F_2) and Fatou's lemma that

$$\lim_{t\to+\infty}\frac{J_\lambda(t\varphi)}{t^2} = \frac{1}{2}\int_\Omega|\nabla\varphi|^2dx - \lim_{t\to+\infty}\frac{1}{t^2}\int_\Omega a(x)F(t\varphi)dx \leq \frac{1}{2}\left(\int_\Omega|\nabla\varphi|^2dx - l\right) < 0.$$

So, $J_\lambda(t\varphi) \to -\infty$ as $t \to +\infty$, then there exists $e \in H_0^1(\Omega)$ with $\|e\| > \rho$ such that $J_\lambda(e) \le -1$. Thus, by (2.7), we have

$$\begin{aligned} I_\lambda(e) &= J_\lambda(e) + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_e e^2 dx \\ &\le -1 + C|K|_\infty^2 \|e\|^4. \end{aligned}$$

Consequently, taking $k^* = C^{-1/2}\|e\|^{-1}$, we deduce that $I_\lambda(e) < 0$ for $|K|_\infty < k^*$. So, the conclusion of the case $K \in L^\infty(\mathbb{R}^3)$ follows from the mountain pass theorem and Lemmas 2.5, 2.7, and 3.4.

While for the case $K \in L^2(\mathbb{R}^3)$, we see that the functional I_λ has a nontrivial critical point by the mountain pass theorem, lemmas 2.4 and 2.7 in a similar way to the case $K \in L^\infty(\mathbb{R}^3)$. Set

$$\mathcal{N}_\lambda = \left\{u \in E_\lambda \setminus \{0\} : I'_\lambda(u) = 0\right\}.$$

Thus, $\mathcal{N}_\lambda$ is nonempty. For any $u \in \mathcal{N}_\lambda$, we have

$$\begin{aligned} 0 = \left\langle I'_\lambda(u), u\right\rangle &= \|u\|_\lambda^2 + \int_{\mathbb{R}^3} K(x)\phi_u u^2 dx - \int_{\mathbb{R}^3} a(x) f(u) u dx \\ &\ge (1 - C_1\epsilon)\|u\|_\lambda^2 - C_2 C_\epsilon \|u\|_\lambda^6. \end{aligned} \tag{3.4}$$

Since $u \neq 0$ whenever $u \in \mathcal{N}_\lambda$, (3.4) implies that the limit point of any sequence on $\mathcal{N}_\lambda$ is different from zero.

We claim that I_λ is bounded from below on $\mathcal{N}_\lambda$, i.e., there exists $\hat{m}_\lambda > 0$ such that

$$I_\lambda(u) \ge -\hat{m}_\lambda, \quad \text{for all } u \in \mathcal{N}_\lambda.$$

Otherwise, there exists $\{u_n\} \subset \mathcal{N}_\lambda$ such that

$$I_\lambda(u_n) < -n, \quad \text{for any } n \in \mathbb{N}. \tag{3.5}$$

It is easy to see that

$$I_\lambda(u_n) \ge \frac{C_3}{4}\|u_n\|_\lambda^2 - C_4\|u_n\|_\lambda^6.$$

This and (3.5) imply that $\|u_n\|_\lambda \to +\infty$ as $n \to \infty$. Let $\omega_n = u_n/\|u_n\|_\lambda$, then there exists $\omega \in E_\lambda$ such that, up to a sequence, (2.9) holds. Note that $I'_\lambda(u_n) = 0$ by $u_n \in \mathcal{N}_\lambda$, as in the proof of Lemma 2.4, we see that $\|u_n\| \to +\infty$ as $n \to \infty$ is impossible. Then, I_λ is bounded from below on $\mathcal{N}_\lambda$. So, we may define

$$\hat{c}_\lambda = \inf\left\{I_\lambda(u) : u \in \mathcal{N}_\lambda\right\}.$$

and $\hat{c}_\lambda \ge -\hat{m}_\lambda$. Let $\{\hat{u}_n\} \subset \mathcal{N}_\lambda$ be such that $I_\lambda(\hat{u}_n) \to \hat{c}_\lambda$, as $n \to \infty$. Following almost the same procedures as the proof of Lemmas 2.4 and 2.7, we can show that $\{\hat{u}_n\}$ is bounded in E_λ and it has a convergent subsequence, strongly converging to $\hat{u} \in E_\lambda \setminus \{0\}$. Thus $I_\lambda(\hat{u}) = \hat{c}_\lambda$ and $I'_\lambda(\hat{u}) = 0$. It then follows from Lemma 3.4 that $\hat{u}$ is a positive ground state of problem $(\mathcal{SP})_\lambda$. The proof is complete. □

Proof of Theorem 1.3. Suppose that $(u, \phi_u) \in E_\lambda \times D^{1,2}(\mathbb{R}^3)$ is a solution of $(\mathcal{SP})_\lambda$. Then, multiplying the first equation in $(\mathcal{SP})_\lambda$ by u and integrating by parts, we obtain

$$\int_{\mathbb{R}^3} \left(|\nabla u|^2 + \lambda V(x)u^2 + K(x)\phi_u u^2 - a(x) f(u) u\right) dx = 0. \tag{3.6}$$

As in (2.23), we have

$$\begin{aligned} k_0 \int_{\mathbb{R}^3} |u|^3 dx \le \int_{\mathbb{R}^3} K(x)|u|^3 dx &= \int_{\mathbb{R}^3} \nabla\phi_u \nabla u dx \\ &\le \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + K(x)\phi_u u^2\right) dx. \end{aligned} \tag{3.7}$$

From (F_1) and (F_2), we see that for any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$f(u)u \le \epsilon u^2 + C_\epsilon |u|^3. \tag{3.8}$$

Then, by the Hölder inequality, we have that for all $\lambda \geq 1$,

$$\int_{\mathbb{R}^3} a(x)f(u)udx \leq \epsilon|a|_\infty \left(\int_{\{V<b\}} u^2dx + \int_{\{V\geq b\}} u^2dx\right) + C_\epsilon|a|_\infty \int_{\mathbb{R}^3} u^3dx$$
$$\leq \epsilon|a|_\infty \max\left\{S^{-1}\left|\{V<b\}\right|^{\frac{2}{3}}, b^{-1}\right\} \|u\|_\lambda^2 + C_\epsilon|a|_\infty \int_{\mathbb{R}^3} u^3dx. \tag{3.9}$$

Let $\epsilon > 0$ small enough such that

$$\epsilon|a|_\infty \max\left\{S^{-1}\left|\{V<b\}\right|^{\frac{2}{3}}, b^{-1}\right\} \leq \frac{1}{2}.$$

Then, (3.9) becomes that

$$\int_{\mathbb{R}^3} a(x)f(u)udx \leq \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2dx + \frac{1}{2}\int_{\mathbb{R}^3} \lambda V(x)u^2dx + C\int_{\mathbb{R}^3} u^3dx. \tag{3.10}$$

Substituting (3.7) and (3.10) into (3.6), we have

$$0 = \int_{\mathbb{R}^3} \left[|\nabla u|^2 + \lambda V(x)u^2 + K(x)\phi_u u^2 - a(x)f(u)u\right]dx$$
$$\geq (k_0 - C)\int_{\mathbb{R}^3} |u|^3dx.$$

Therefore, if $k_0 > C$, then u must be zero. The proof is complete. □

Proof of Theorem 1.4. It follows from the conditions (F_1), (F_3), and (F_4) that there exist constants $C_1 > 0$ and $C_2 > 0$ such that

$$F(s) \geq C_1 s^4 - C_2 s^2, \quad \text{for all } s > 0.$$

Similar to the proof of [Ref. 42, Theorem 1.2], there exists a $(C)_c$ sequence $\{u_n\} \subset E_\lambda$ such that

$$I_\lambda(u_n) \to c_\lambda \quad \text{and} \quad I'_\lambda(u_n) \to 0 \text{ in } E^*_\lambda, \tag{3.11}$$

where

$$c_\lambda = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_\lambda(\gamma(t)),$$

with $\Gamma = \left\{\gamma \in C\left([0,1], E^*_\lambda\right) : \gamma(0) = 0, I_\lambda(e) < 0\right\}$. By (V_3), and noting that $e \in H^1_0(\Omega)$, we can easily see that there exists a constant $C_0 > 0$, independent of λ, such that for all $\lambda > 0$,

$$c_\lambda = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_\lambda(\gamma(t)) \leq \max_{t\in[0,1]} I_\lambda(te) \leq C_0. \tag{3.12}$$

It then follows from Lemmas 2.6 and 2.8 that $u_n \to u$ in E_λ for all $\lambda \geq \overline{\Lambda}$. Thus, u is a nontrivial critical point of $(\mathcal{SP})_\lambda$. Set

$$\mathcal{N}_\lambda = \left\{u \in E_\lambda \setminus \{0\} : I'_\lambda(u) = 0\right\}.$$

So, $\mathcal{N}_\lambda$ is nonempty. For any $u \in \mathcal{N}_\lambda$, we have

$$0 = \left\langle I'_\lambda(u), u\right\rangle = \|u\|_\lambda^2 + \int_{\mathbb{R}^3} K(x)\phi_u u^2dx - \int_{\mathbb{R}^3} a(x)f(u)udx$$
$$\geq (1 - C_1\epsilon)\|u\|_\lambda^2 - C_2C_\epsilon\|u\|_\lambda^6,$$

which implies that the limit point of any sequence on $\mathcal{N}_\lambda$ is different from zero.

We claim that I_λ is bounded from below on $\mathcal{N}_\lambda$. It follows from $u \in \mathcal{N}_\lambda$ and (F_4) that

$$I_\lambda(u) = \frac{1}{4}\|u\|_\lambda^2 - \int_{\mathbb{R}^3} a(x)\left[F(u) - \frac{1}{4}f(u)u\right]dx$$
$$\geq \frac{1}{4}\|u\|_\lambda^2 > 0.$$

So, I_λ is bounded from below on $\mathcal{N}_\lambda$. Hence, we may define

$$\tilde{c}_\lambda = \inf\left\{I_\lambda(u) : u \in \mathcal{N}_\lambda\right\}.$$

It is easy to see that $\tilde{c}_\lambda > 0$. Let $\{\tilde{u}_n\} \subset \mathcal{N}_\lambda$ be such that $I_\lambda(\tilde{u}_n) \to \tilde{c}_\lambda$, as $n \to \infty$. Following almost the same procedures as the proof of Lemmas 2.6 and 2.8, we can show that $\{\tilde{u}_n\}$ is bounded in E_λ and it has a convergent subsequence, strongly converging to $\tilde{u} \in E_\lambda \setminus \{0\}$. Thus $I_\lambda(\tilde{u}) = \tilde{c}_\lambda$ and $I'_\lambda(\tilde{u}) = 0$. It then follows from Lemma 3.4 that $\tilde{u}$ is a positive ground state solution of problem $(\mathcal{SP})_\lambda$. The proof is complete. □

IV. ASYMPTOTIC BEHAVIOR OF POSITIVE GROUND STATE SOLUTIONS

In this section, we investigate the asymptotic behavior of positive ground state solutions for system $(\mathcal{SP})_\lambda$ and give the proof of Theorem 1.5.

Proof of Theorem 1.5. We follow the argument in Ref. 4 (or see Ref. 42). For any sequence $\lambda_n \to \infty$, let $u_n := u_{\lambda_n}$ be the positive ground state solution of system $(\mathcal{SP})_{\lambda_n}$ obtained in Theorem 1.4. By (A_2) and (F_4), we have

$$\begin{aligned}\tilde{c}_{\lambda_n} &= \frac{1}{4}\|u_n\|^2_{\lambda_n} + \frac{1}{4}\int_{\mathbb{R}^3} a(x)\,[f(u_n)u_n - 4F(u_n)]\,dx \\ &\geq \frac{1}{4}\|u_n\|^2_{\lambda_n}.\end{aligned} \tag{4.1}$$

Combining (4.1) with (3.12), one has

$$\sup_{n\geq 1}\|u_n\|^2_{\lambda_n} \leq 4C_0, \tag{4.2}$$

where the constant C_0 is independent of λ_n. Consequently, up to a subsequence, we may assume that for some nonnegative $u_0 \in E$,

$$\begin{cases} u_n \rightharpoonup u_0 \text{ in } E, \\ u_n \to u_0 \text{ in } L^s_{loc}(\mathbb{R}^3) \text{ for all } s \in [1,6), \\ u_n \to u_0 \text{ a.e. in } \mathbb{R}^3. \end{cases} \tag{4.3}$$

By Fatou's lemma and (4.2), we obtain

$$\int_{\mathbb{R}^3} V(x)u_0^2dx \leq \liminf_{n\to\infty}\int_{\mathbb{R}^3} V(x)u_n^2dx \leq \liminf_{n\to\infty}\frac{\|u_n\|^2_{\lambda_n}}{\lambda_n} = 0,$$

thus $u_0 = 0$ a.e. in $\mathbb{R}^3 \setminus V^{-1}(0)$, and $u_0 \in H_0^1(\Omega)$ by the condition (V_3). Now for any $\varphi \in C_0^\infty(\Omega)$, since $\langle I'_{\lambda_n}(u_n), \varphi\rangle = 0$, it is easy to check that

$$\int_{\mathbb{R}^3}\nabla u_0\nabla\varphi dx + \int_{\mathbb{R}^3} K(x)\phi_{u_0}u_0\varphi dx = \int_{\mathbb{R}^3} a(x)f(u_0)\varphi dx,$$

that is, u_0 is a weak nonnegative solution of $(\mathcal{SP})_\infty$ by the density of $C_0^\infty(\Omega)$ in $H_0^1(\Omega)$.

Next, we show that $u_n \to u_0$ in $L^s(\mathbb{R}^3)$ for $2 < s < 6$. Otherwise, by Lions vanishing lemma[22] there exist $\delta > 0$, $r > 0$ and $y_n \in \mathbb{R}^3$ such that

$$\int_{B_r(y_n)} |u_n - u_0|^2dx \geq \delta.$$

Moreover, $|y_n| \xrightarrow{n} \infty$, so $|B_r(y_n) \cap \{V < b\}| \xrightarrow{n} 0$. Then, by the Hölder inequality we have

$$\int_{B_r(y_n)\cap\{V<b\}} |u_n - u_0|^2dx \xrightarrow{n} 0.$$

Consequently,

$$\begin{aligned}\|u_n\|^2_{\lambda_n} &\geq \lambda_n b\int_{B_r(y_n)\cap\{V\geq b\}} u_n^2dx = \lambda_n b\int_{B_r(y_n)\cap\{V\geq b\}} |u_n - u_0|^2dx \\ &= \lambda_n b\left(\int_{B_r(y_n)} |u_n - u_0|^2dx - \int_{B_r(y_n)\cap\{V<b\}} |u_n - u_0|^2dx\right) \to \infty,\end{aligned}$$

as $n \to \infty$, which contradicts (4.2).

To complete the proof, it suffices to show that $u_n \to u_0$ in E. It follows from $\langle I'_{\lambda_n}(u_n), u_n\rangle = \langle I'_{\lambda_n}(u_n), u_0\rangle = 0$ that

$$\|u_n\|^2_{\lambda_n} + \int_{\mathbb{R}^3} K(x)\phi_{u_n}u_n^2 dx = \int_{\mathbb{R}^3} a(x)f(u_n)u_n dx, \tag{4.4}$$

$$\langle u_n, u_0\rangle_{\lambda_n} + \int_{\mathbb{R}^3} K(x)\phi_{u_n}u_n u_0 dx = \int_{\mathbb{R}^3} a(x)f(u_n)u_0 dx. \tag{4.5}$$

We claim that

$$\int_{\mathbb{R}^3} (a(x)f(u_n)u_n - a(x)f(u_n)u_0)\, dx \xrightarrow{n} 0 \tag{4.6}$$

and

$$\int_{\mathbb{R}^3} \left(K(x)\phi_{u_n}u_n^2 - K(x)\phi_{u_n}u_n u_0\right) dx \xrightarrow{n} 0. \tag{4.7}$$

Indeed, by a standard procedure, we can show that (4.6) holds. On the other hand, if $K \in L^\infty(\mathbb{R}^3)$, from the Hölder inequality and $u_n \to u_0$ in $L^3(\mathbb{R}^3)$ we conclude that

$$\int_{\mathbb{R}^3} \left(K(x)\phi_{u_n}u_n^2 - K(x)\phi_{u_n}u_n u_0\right) dx \le |K|_\infty |\phi_{u_n}|_6 |u_n|_2 |u_n - u_0|_3 \xrightarrow{n} 0,$$

while for $K \in L^2(\mathbb{R}^3)$, we obtain (4.7) in a similar way to (2.33).

By (4.4)–(4.7), we yield that

$$\lim_{n\to\infty} \|u_n\|^2_{\lambda_n} = \lim_{n\to\infty} \langle u_n, u_0\rangle_{\lambda_n} = \lim_{n\to\infty} \langle u_n, u_0\rangle = \|u_0\|^2.$$

On the other hand, weakly lower semi-continuity of norm implies that

$$\|u_0\|^2 \le \liminf_{n\to\infty} \|u_n\|^2 \le \liminf_{n\to\infty} \|u_n\|^2_{\lambda_n},$$

hence $u_n \to u_0$ in E. From (4.4) and the fact that $u_n \ne 0$, we deduce that for n large,

$$\|u_n\|^2 \le \|u_n\|^2_{\lambda_n} \le C\|u_n\|^6,$$

which implies that $u_0 \ne 0$. By the strong maximum principle, u_0 is a positive solution of $(\mathcal{SP})_\infty$. This completes the proof. □

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