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RESEARCH ARTICLE

Exponential input-to-state stability of recurrent neural networks with multiple time-varying delays

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Abstract In this paper, input-to-state stability problems for a class of recurrent neural networks model with multiple time-varying delays are concerned with. By utilizing the Lyapunov–Krasovskii functional method and linear matrix inequalities techniques, some sufficient conditions ensuring the exponential input-to-state stability of delayed network systems are firstly obtained. Two numerical examples and its simulations are given to illustrate the efficiency of the derived results.

Keywords Exponential input-to-state stability (exp-ISS) · Recurrent neural networks · Multiple time-varying delays

Introduction

The latest few decades have witnessed the use of neural networks in many real-world applications and have offered an attractive paradigm for a broad range of adaptive complex systems. Due to its wide applications in various areas such as pattern classification, associative memory, parallel computation, optimization, moving object speed detection and so on, recurrent neural networks (RNNs) have been extensively studied by researchers in recent years (see, e.g., (Hopfield 1984; Hopfield and Tank 1986; Grujiá and Michel 1991; Matsouka 1992; Arik 2000; Ensari and Arik 2005; Zhang et al. 2008; Wu et al. 2008, 2010; Huang et al. 2012; Huang and Feng 2009; Ahn 2010a; Liu and Cao 2010; Ahn 2010b, c, 2011a, b, 2012a, b, c; Sanchez and Perez 1999; Zhu and Shen 2012) and references therein). Since time-delay is unavoidably encountered in implementation of RNNs and is frequently a source of oscillation and instability, the stability of delayed neural networks has become a topic of great theoretical and practical importance, and many interesting results on stability in the Lyapunov sense have been derived (see also e.g. (Arik 2000; Ensari and Arik 2005; Zhang et al. 2008; Wu et al. 2008, 2010; Huang et al. 2012) and references therein).

It is well known that dynamical behaviors of neural networks are often affected by disturbances such as control inputs, external perturbations or errors on observation. Some methods, e.g., H_∞ control in Huang and Feng 2009 and Ahn 2010a and state estimation in Liu and Cao 2010and Ahn 2012a), were given to discuss the influence of disturbances for the dynamical behaviors of the networks. Some dynamical properties such as robustness (see, e.g.,Wu et al. 2010; Huang et al. 2012; Ahn 2012b), passivity (see Ahn 2011b, 2012c), input-to-state stability (see Sanchez and Perez 1999; Ahn 2010b, 2011a, b, 2010c; Zhu and Shen 2012) also are investigated for the networks with disturbances. Among these properties, the input-to-state stability (ISS) are widely accepted as an important tool to check robust stability since the ISS properties imply not only that the unperturbed system is asymptotically stable in the Lyapunov sense but also that its behavior remains

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bounded when its inputs are bounded. It also offers an effective way to tackle the stabilization of nonlinear control in the presence of various uncertainties arising from control engineering applications. The concept of ISS is firstly introduced in nonlinear control systems by Sontag (1989, 1990), and the various extension of ISS such as integral input-to-state stability (i-ISS), finite-time input-to-state stability and exponential input-to-state stability (exp-ISS) have been investigated for nonlinear control systems (see also Sontag and Wang 1995; Angeli et al. 2000; Jiang and Wang 2001; Hong et al. 2010; Yang and Zhou 2012a, 2012b)). Due to these research background, the ISS properties of neural networks are investigated in recent years. For example, Sanchez and Perez (1999) firstly investigated the ISS properties and gave some matrix norm conditions on ISS for RNNs. Ahn (2010b) proposed passivity-based learning law to investigate the ISS for a class of switched Hopfield neural networks with time delay. Also in Ahn (2011a), by employing a suitable Lyapunov function, a new sufficient condition is derived to guarantee ISS of Takagi-Sugeno fuzzy Hopfield neural networks with time-delay. Moreover, some LMI sufficient conditions have been proposed to guarantee the ISS by utilizing Lyapunov function method in Ahn (2011b). In Zhu and Shen (2012), two new results on input-to-state stability of recurrent neural networks with time-varying delays are given. However, to the best of the authors knowledge, there are few results on the exp-ISS for RNNs with multiple time-varying delays, while it attains a much faster convergence rate.

Motivated by the above discussions, we firstly study exp-ISS properties of RNNs with multiple time-varying delays in this paper. By using Lyapunov–Krasovskii functional technique, two sufficient conditions ensuring exp-ISS are given in terms of LMIs for the delayed neural networks and the exponential convergence rate is estimated. We also provide two illustrative examples to demonstrate the effectiveness of the proposed results.

Mathematical model and preliminaries

Let R^n denote the n—dimensional Euclidean space, $R^{n\times n}$ be the set of all $n \times n$ real matrices , I denote the element matrix. $|\cdot|$ denotes the usual Euclidean norm, or the induced Euclidean norm of a matrix. Let $B^T, B^{-1}, \lambda_{max}(B)$, $\lambda_{min}(B)$ and $\|B\|_2 = \sqrt{\lambda_{max}(B^TB)}$ denote the transpose, the inverse, the largest eigenvalue, the smallest eigenvalue, and the Euclidean norm of a square matrix B, respectively. The notation $P > 0 (\geq 0)$ means that P is real symmetric and positive definite (positive semi-definite). The notion $X > Y$ $(X \geq Y)$, where X and Y are symmetric matrices, means that $X - Y$ is positive define (positive semi-definite).

Let $\tau > 0$. Denote by $C([-\tau, 0], R^n)$ (or simply denote by C) the family continuous functions mapping the interval $[-\tau,0]$ into R^n with the norm of an element in C by $\|\phi\|_\tau = \sup_{-\tau\leq\vartheta\leq 0}|\phi(\vartheta)|$. The set of all measurable locally essentially bounded functions $u{:}R^+\to R^n$, endowed with (essential) supremum norm $\|u\|_\infty = sup\{|u(t)|, t\geq 0\}$, is denoted by L_∞^m. We recall that a function $\gamma{:}R^+\to R^+$ is a K—function if it is continuous, strictly increasing, and $\gamma(0) = 0$; it will be a K_∞*—function* if it is a K*—function* and also $\gamma(s)\to\infty$ as $s\to\infty$. A function $\beta{:}R^+\times R^+\to R^+$ is a KL*—function* if for each fixed $t\geq 0$ the function $\beta(\cdot, t)$ is a K*—function*, and for each fixed $s \geq 0$, it is decreasing to zero as $t\to\infty$.

In this paper, we consider the following recurrent neural networks with multiple time-varying delays:

$$\frac{dx_i(t)}{dt} = -c_ix_i(t) + \sum_{j=1}^{n} a_{ij}f_j(x_j(t)) + \sum_{k=1}^{N}\sum_{j=1}^{n} b_{ij}^{(k)} f_j(x_j(t-\tau_{kj}(t))) + u_i(t), \tag{1}$$

where $i = 1, 2, \ldots, n$, $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$ is the neuron state vector, n is the number of neurons, $C = diag\{c_1, c_2, \ldots, c_n\}$ with $c_i > 0$, $A = (a_{ij})_{n\times n} \in R^{n\times n}$ and $B_k = (b_{ij}^{(k)})_{n\times n}$ are the connection weight matrix and delayed connection weight matrix, respectively, N denotes the number of delayed connection matrices,time delay $\tau_{kj}(t) \geq 0$, $f(x(t)) = (f_1(x_1(t)), f_2(x_2(t)), \ldots, f_n(x_n(t)))^T$, $f(x(t-\tau_k(t))) = (f_1(x_1(t-\tau_{k1}(t))), f_2(x_2(t-\tau_{k2}(t))), \ldots, f_n(x_n(t-\tau_{kn}(t))))^T$, $k = 1, 2, \ldots, N$, $j = 1, 2, \ldots, n$, and $u(t) = (u_1(t), u_2(t), \ldots, u_n(t))^T$ denotes the external input signal to neurons. For convenience, (1) can be rewritten as

$$\frac{dx(t)}{dt} = -Cx(t) + Af(x(t)) + \sum_{k=1}^{N} B_kf(x(t-\tau_k(t))) + u(t). \tag{2}$$

In the network system, the external input signal $u(t)$ is likely to time-varying even unknown, but its varying rang can be estimated. In this case, it is interesting to investigate the properties of ISS or exp-ISS defined below.

Definition 2.1 The delayed recurrent neural networks of (2) is said to be input-to-state stable if there exists a KL—function β: $R^+\times R^+\to R^+$ and a K—function $\gamma(\cdot)$ such that the solution $x(t;\xi, u(t))$ satisfies

$$|x(t;\xi,u(t))| \leq \beta(\|\xi\|_\tau, t) + \gamma(\|u\|_\infty), \tag{3}$$

for any $\xi \in C$, $u(t) \in L_\infty^m$ and $t \geq 0$.

Definition 2.2 The delayed recurrent neural networks of (2) is said to be exp-ISS, if there exists a $\lambda > 0$ and two K—functions β, γ such that the solution $x(t;\xi, u(t))$ satisfies

$$|x(t;\xi,u(t))| \le \beta(\|\xi\|_\tau)e^{-\lambda t} + \gamma(\|u\|_\infty), \tag{4}$$

for any $\xi \in C$, $u(t) \in L^n_\infty$ and $t \ge 0$.

Clearly, exp-ISS implies ISS. To obtain the exp-ISS properties, we need the following basic lemmas involving linear matrix inequalites (LMIs).

Lemma 2.3 (Zhang et al. 2008) *Let X and Y be two real vectors with appropriate dimensions, and let Q and Π be two matrices with appropriate dimensions, where $Q > 0$. Then, for any two positive constants $m > 0$ and $l > 0$, the following inequality holds*

$$-mX^TQX + 2lX^T\Pi Y \le l^2Y^T\Pi^T(mQ)^{-1}\Pi Y.$$

Lemma 2.4 (Horn and Johnson 1991) *Given any matrix X, Y, and Λ with appropriate dimensions such that $\Lambda = \Lambda^T > 0$ and any scalar $\varepsilon > 0$, then*

$$X^TY + Y^TX \le \varepsilon X^T\Lambda X + \frac{1}{\varepsilon}Y^T\Lambda^{-1}Y.$$

Main results

In this section, two sufficient conditions ensuring the exponential input-to-state stability property are presented for the neural network system (2).

In order to obtain our main results, we always suppose that

(A1). there exists $h_j > 0$ such that $\forall\theta, \rho \in R, 0 \le \frac{f_j(\theta)-f_j(\rho)}{\theta-\rho} \le h_j$, and $f_j(0) = 0$, $j = 1, 2, \ldots, n$.

(A2). $0 \le \tau_{kj}(t) \le \tau_j(t) < \tau, 0 \le \dot{\tau}_{kj}(t) \le \mu_{kj} < 1, k = 1,2, \ldots, N, j = 1,2,\ldots,n.$

The above assumption (A 1) can be satisfied for some typical sigmoid activation functions, for example, $f_i(x_i) = \frac{a_i}{1+e^{-b_ix_i}}$ given in (Matsouka 1992).

Theorem 1 *Let (A1) and (A2) hold. Then the delayed neural network of (2) is exponentially input-to-state stable if there exist $L = diag\{l_1, l_2, \ldots, l_n\} > 0$ and $Q_k = diag\{q_{k1}, q_{k2}, \ldots, q_{kn}\} > 0$, $k = 1, 2, \ldots, N$, satisfy*

$$\begin{aligned} G :=& LA + A^TL - 2LC\Delta^{-1} \\ &+ \sum_{k=1}^{N}\left(\frac{1}{\gamma_k}LB_kQ_k^{-1}B_k^TL + Q_k\right) < 0, \end{aligned} \tag{5}$$

where $\gamma_k = \min\limits_{1\le j\le n}(1-\mu_{kj}), \Delta = diag\{h_1, h_2, \ldots, h_n\}$.

Proof Let $P_k = diag\{p_{k1}, p_{k2}, \ldots, p_{kn}\}$ satisfy

$$P_k \ge 2NB_k^TB_k \ge 0. \tag{6}$$

From (5), we choose a constant $\alpha > 0$ and a small enough $\varepsilon > 0$ satisfying

$$\alpha G + A^TA + \sum_{k=1}^{N}\frac{1}{1-\mu}P_k + \varepsilon\alpha I < 0, \tag{7}$$

and

$$-\frac{1}{2} + \epsilon + \epsilon\rho + \sum_{k=1}^{N}(\alpha q_{ki} + \frac{p_{kj}}{1-\mu})h_i^2\epsilon\tau e^{\epsilon\tau} \le 0, \tag{8}$$

where

$$\rho := \max_{1\le i\le n}(c_i + \alpha l_ih_i) > 0,$$
$$0 < \mu := \sup_{1\le k\le N}\max_{1\le j\le n}(\mu_{kj}) < 1.$$

□

Construct a positive definite and radically unbound Lyapunov functional

$$\begin{aligned} V(x(t),t) =& x^T(t)Cx(t) + 2\alpha\sum_{i=1}^{n}l_i\int_0^{x_i(t)}f_i(s)ds \\ &+ \sum_{k=1}^{N}\sum_{j=1}^{n}\left[\frac{1}{1-\mu}\int_{t-\tau_{kj}(t)}^{t}p_{kj}f_j^2(x_j(s))ds \right. \\ &\left. +\alpha\int_{t-\tau_{kj}(t)}^{t}q_{kj}f_j^2(x_j(s))ds\right]. \end{aligned} \tag{9}$$

The derivative of (9) along the trajectories of (2) is obtained as follows:

$$\begin{aligned} \dot{V}(x(t),t)\mid_{(2)} \le& 2x^T(t)C[-Cx(t) + Af(x(t)) \\ &+ \sum_{k=1}^{N}B_kf(x(t-\tau_k(t))) + u(t)] \\ &+ 2\alpha f^T(x(t))L[-Cx(t) + Af(x(t)) \\ &+ \sum_{k=1}^{N}B_kf(x(t-\tau_k(t))) + u(t)] \\ &+ \sum_{k=1}^{N}[\frac{1}{1-\mu}f^T(x(t))P_kf(x(t)) \\ &- f^T(x(t-\tau_k(t)))P_kf(x(t-\tau_k(t))) \\ &+ \alpha f^T(x(t))Q_kf(x(t)) \\ &- \alpha\gamma_kf^T(x(t-\tau_k(t)))Q_kf(x(t-\tau_k(t)))]. \end{aligned} \tag{10}$$

From Lemma 2.3 and (6), we have

$$
\begin{aligned}
&-x^T(t)CCx(t)+2x^T(t)CAf(x(t))\\
&\quad\le f^T(x(t))A^TAf(x(t)),\\
&\quad-\frac{1}{N}x^T(t)CCx(t)+2x^T(t)CB_kf(x(t-\tau_k(t)))\\
&\quad\le 2Nf^T(x(t-\tau_k(t)))B_k^TB_kf(x(t-\tau_k(t)))\\
&\quad\le f^T(x(t-\tau_k(t)))P_kf(x(t-\tau_k(t))),\\
&\quad-\alpha\gamma_kf^T(x(t-\tau_k(t)))Q_kf(x(t-\tau_k(t)))\\
&\quad+2\alpha f^T(x(t))LB_kf(x(t))\\
&\quad\le\frac{\alpha}{\gamma_k}f^T(x(t))LB_kQ_k^{-1}B_k^TLf(x(t)).
\end{aligned}\tag{11}
$$

According to (A1), then

$$-\alpha f^T(x(t))LCx(t)\le-\alpha f^T(x(t))LC\Delta^{-1}f(x(t)).\tag{12}$$

Using Lemma 2.4 with $\varepsilon>0$ given in (7) and (8), we have

$$2x^T(t)Cu(t)\le\varepsilon x^T(t)x(t)+\frac{u^T(t)CCu(t)}{\varepsilon},\tag{13}$$

$$2f^T(x(t))Lu(t)\le\varepsilon f^T(x(t))f(x(t))+\frac{u^T(t)L^2u(t)}{\varepsilon}.\tag{14}$$

Substituting (11)–(14) into (10), we obtain

$$
\begin{aligned}
\dot V(x(t),t)\,|_{(2)}\le{}&f^T(x(t))[A^TA+\sum_{k=1}^N\frac{1}{1-\mu}P_k\\
&+\sum_{k=1}^N\alpha Q_k+2\alpha LA+\sum_{k=1}^N\frac{\alpha}{\gamma_k}LB_kQ_k^{-1}B_k^TL\\
&-2\alpha LC\Delta^{-1}+\varepsilon\alpha I]f(x(t))\\
&-(\frac{1}{2}-\epsilon)x^T(t)x(t)+\frac{1}{\varepsilon}u^T(t)(C^2+\alpha L^2)u(t)\\
&-\eta^*|f(x(t))|^2-(\frac{1}{2}-\epsilon)|x(t)|^2+\xi^*|u(t)|^2,
\end{aligned}\tag{15}
$$

where $\xi^*:=\max_{1\le i\le n}\frac{1}{\varepsilon}(c_i^2+\alpha l_i^2),\eta^*:=-\lambda_{max}(\alpha G+A^TA+\sum_{k=1}^N\frac{1}{1-\mu}P_k+\varepsilon\alpha I)$, and ξ^*, $\eta^*>0$ from (7).

Then, deriving the derivative of $e^{\varepsilon t}V(x(t),t)$, we get

$$
\begin{aligned}
\frac{d(e^{\epsilon t}V(x(t),t))}{dt}\,|_{(2)}={}&\epsilon e^{\epsilon t}V(x(t),t)+e^{\epsilon t}\dot V(x(t),t)\,|_{(2)}\\
\le{}&\epsilon e^{\epsilon t}\{x^T(t)Cx(t)+2\alpha\sum_{i=1}^n l_i\int_0^{x_i(t)}f_i(s)ds\\
&+\sum_{k=1}^N\sum_{j=1}^n[\frac{1}{1-\mu}\int_{t-\tau_{kj}(t)}^t p_{kj}f_j^2(x_j(s))ds\\
&+\alpha\int_{t-\tau_{kj}(t)}^t q_{kj}f_j^2(x_j(s))ds]\}\\
&+e^{\epsilon t}[-\eta^*|f(x(t))|^2-(\frac{1}{2}-\epsilon)|x(t)|^2+\xi^*|u(t)|^2].
\end{aligned}
$$

Since

$$2\sum_{i=1}^n l_i\int_0^{x_i(t)}f_i(s)ds\le x^T(t)L\Delta x(t),$$

then

$$
\begin{aligned}
\frac{d(e^{\epsilon t}V(x(t),t))}{dt}\,|_{(2)}\le{}&e^{\epsilon t}[-\eta^*|f(x(t))|^2\\
&+\sum_{i=1}^n(-(\eta^*-\epsilon)+\epsilon(c_i+\alpha l_ih_i))|x_i(t)|^2\\
&+\sum_{k=1}^N\sum_{j=1}^n(\alpha q_{kj}+\frac{p_{kj}}{1-\mu})h_j^2\varepsilon\int_{t-\tau_j(t)}^t|x_j(s))|^2ds)\\
&+\xi^*|u(t)|^2].
\end{aligned}
$$

when $t\ge\tau$, integrating both sides of (16), we can obtain

$$
\begin{aligned}
e^{\epsilon t}V(x(t),t)\le{}&V(x(0),0)-\eta^*\int_0^t e^{\epsilon s}|f(x(s))|^2ds\\
&-(\frac{1}{2}-\epsilon-\epsilon\rho)\int_0^t e^{\epsilon s}|x(s)|^2ds\\
&+\sum_{k=1}^N\sum_{j=1}^n(\alpha q_{kj}+\frac{p_{kj}}{1-\mu})h_j^2\epsilon\int_0^t e^{\epsilon s}\int_{s-\tau}^s|x_j(\zeta)|^2d\zeta ds\\
&+\xi^*\int_0^t e^{\epsilon s}|u(s)|^2ds.
\end{aligned}\tag{16}
$$

By exchanging the integrals, we have

$$
\begin{aligned}
&\sum_{k=1}^N\sum_{j=1}^n(\alpha q_{kj}+\frac{p_{kj}}{1-\mu})h_j^2\epsilon\int_0^t e^{\epsilon s}\int_{s-\tau}^s|x_j(\zeta)|^2d\zeta ds\\
&=\sum_{k=1}^N\sum_{j=1}^n(\alpha q_{kj}+\frac{p_{kj}}{1-\mu})h_j^2\epsilon\\
&\times\int_{-\tau}^t|x_j(\zeta)|^2\int_{max(0,\zeta)}^{min(t,\zeta+\tau)}e^{\epsilon s}dsd\zeta\\
&\le\sum_{k=1}^N\sum_{j=1}^n(\alpha q_{kj}+\frac{p_{kj}}{1-\mu})h_i^2\epsilon\int_{-\tau}^t\tau e^{\epsilon(\zeta+\tau)}|x_j(\zeta)|^2d\zeta\\
&\le\sum_{k=1}^N\sum_{j=1}^n[(\alpha q_{kj}+\frac{p_{kj}}{1-\mu})h_j^2\epsilon\tau e^{\epsilon\tau}\int_{-\tau}^0|x_j(s)|^2ds\\
&\quad+(\alpha q_{kj}+\frac{p_{kj}}{1-\mu})h_j^2\epsilon\tau e^{\epsilon\tau}\int_0^t e^{\epsilon s}|x_j(s)|^2ds].
\end{aligned}\tag{17}
$$

Substituting (17) into (16), when $t \geq \tau$, then we have

$$
\begin{aligned}
e^{\epsilon t}V(x(t),t) \leq & V(x(0),0) - \eta^* \int_0^t e^{\epsilon s}|f(x(s))|^2 ds \\
& + \sum_{i=1}^n [-\frac{1}{2} + \epsilon + \epsilon\rho + \sum_{k=1}^N (\alpha q_{ki} + \frac{p_{ki}}{1-\mu}) h_i^2 \epsilon\tau e^{\epsilon\tau}] \\
& \times \int_0^t e^{\epsilon s}|x_i(s)|^2 ds \\
& + \sum_{k=1}^N \sum_{j=1}^n (\alpha q_{kj} + \frac{p_{kj}}{1-\mu}) \times h_j^2 \epsilon\tau e^{\epsilon\tau} \int_{-\tau}^0 |x_j(s)|^2 ds \\
& + \xi^* \int_0^t e^{\epsilon s}|u(s)|^2 ds.
\end{aligned}
$$

From $\eta^* > 0$ and (8),

$$
\begin{aligned}
V(x(t),t) \leq & [\sum_{k=1}^N \sum_{j=1}^n (\alpha q_{kj} + \frac{p_{kj}}{1-\mu}) h_j^2 \epsilon\tau e^{\epsilon\tau} \int_{-\tau}^0 |x_j(s)|^2 ds \\
& + x^T(0)Cx(0) + 2\alpha \sum_{i=1}^n l_i \int_0^{x_i(0)} f_i(s) ds \\
& + \sum_{k=1}^N \sum_{j=1}^n \frac{1}{1-\mu} \int_{-\tau_{kj}(0)}^0 p_{kj} f_j^2(x_j(s)) ds \\
& + \sum_{k=1}^N \sum_{j=1}^n \alpha \int_{-\tau_{kj}(0)}^0 q_{kj} f_j^2(x_j(s)) ds \\
& + \sum_{i=1}^n \xi^* \int_0^t e^{\epsilon s}|u_i(s)|^2 ds] e^{-\epsilon t} \\
\leq & [\sum_{k=1}^N \sum_{j=1}^n (\alpha q_{kj} + \frac{p_{kj}}{1-\mu}) h_j^2 \epsilon\tau e^{\epsilon\tau} \int_{-\tau}^0 |x_j(s)|^2 ds \\
& + \sum_{i=1}^n \rho_i |x_i(0)|^2 \\
& + \sum_{k=1}^N \sum_{j=1}^n (\alpha q_{kj} + \frac{p_{kj}}{1-\mu}) h_j^2 \int_{-\tau_j(0)}^0 |x_j(s)|^2 ds] e^{-\epsilon t} \\
& + \frac{n\xi^*}{\epsilon} \|u\|_\infty^2.
\end{aligned}
$$

According to the definition of $V(x(t), t)$, then we have

$$
\begin{aligned}
|x(t)|^2 \leq & [\frac{\rho}{c^*} |x(0)|^2 \\
& + \frac{1}{c^*} \sum_{k=1}^N \sum_{j=1}^n (\alpha q_{kj} + \frac{p_{kj}}{1-\mu}) h_j^2 \epsilon\tau e^{\epsilon\tau} \int_{-\tau}^0 |x_j(s)|^2 ds \\
& + \frac{1}{c^*} \sum_{k=1}^N \sum_{j=1}^n (\alpha q_{kj} + \frac{p_{kj}}{1-\mu}) h_j^2 \int_{-\tau_j(0)}^0 |x_j(s)|^2 ds] e^{-\epsilon t} \\
& + \frac{n\xi^*}{c^*\epsilon} \|u\|_\infty^2,
\end{aligned}
$$

where $c^* = \min_{1 \leq i \leq n} c_i$. This implies that

$$
\begin{aligned}
|x(t)| \leq & e^{-\frac{\epsilon}{2}t} [\frac{\rho}{c^*} |x(0)|^2 \\
& + \frac{1}{c^*} \sum_{k=1}^N \sum_{j=1}^n (\alpha q_{kj} + \frac{p_{kj}}{1-\mu}) h_j^2 \epsilon\tau e^{\epsilon\tau} \int_{-\tau}^0 |x_j(s)|^2 ds \\
& + \frac{1}{c^*} \sum_{k=1}^N \sum_{j=1}^n (\alpha q_{kj} + \frac{p_{kj}}{1-\mu}) h_j^2 \int_{-\tau_j(0)}^0 |x_j(s)|^2 ds]^{-\frac{1}{2}} \\
& + \sqrt{\frac{n\xi^*}{c^*\epsilon}} \|u\|_\infty.
\end{aligned}
$$

Defining the following functions:

$$
\begin{aligned}
\beta(r) &= \sqrt{\frac{1}{c^*} \sum_{k=1}^N \sum_{j=1}^n h_j^2 (\alpha q_{kj} + \frac{p_{kj}}{1-\mu})(\epsilon\tau e^{\epsilon\tau} + 1) + \frac{\rho r}{c^*}}, \\
\gamma(r) &= r\sqrt{\frac{n\xi^*}{c^*\epsilon}}.
\end{aligned}
$$

Clearly, $\beta(\cdot), \gamma(\cdot)$ are K—functions. Let $\lambda = \frac{\varepsilon}{2}$, then we have

$$
|x(t)| \leq \beta(\|\xi\|_\tau) e^{-\lambda t} + \gamma(\|u\|_\infty). \tag{18}
$$

From the definition, the delayed recurrent neural network (2) is exponentially input-to-state stable. The proof is complete.

Remark 1. Theorem 1 reduces to sufficient condition ensuring exponential stability of the zero solution for delayed recurrent neural network (2) when the input $u(t) = 0$.

Remark 2. Recently, some results on ISS or IOSS were obtained in (Sanchez and Perez 1999; Ahn 2010b, 2011a, b, 2010c). However, these results were restricted to non-delay or constant delay. In contrast to the results (Sanchez and Perez 1999; Ahn 2010b, 2011a, b, 2010c), we consider dynamical neural networks with multiple time-varying delays which may be more accurate to describe the evolutionary process in some real systems. Furthermore, we firstly investigates the exp-ISS of the RNNs, which attains a much faster convergence rate. In fact, an estimation of the exponential convergence rate ϵ can be given by (7) and (8).

Theorem 2: Let (A1) and (A2) hold. Then delayed neural network of (2) is exp-ISS if there exist positive diagonal matrices $D = diag\{d_1, d_2, \ldots, d_n\}, P_k = diag\{p_{k1}, p_{k2}, \ldots, p_{kn}\}, Q_k = diag\{q_{k1}, q_{k2}, \ldots, q_{kn}\}, k = 1, 2, \ldots, N,$ such that

$$
c_i - 2\sum_{k=1}^N p_{ki} > 0, i = 1, 2, \ldots, n, \tag{19}
$$

$$
\begin{aligned}
\Omega :=& DA + A^T D - 2DC\Delta^{-1} \\
&+ \sum_{k=1}^{N} \Big(\frac{1}{\gamma_k} DB_k P_k^{-1} B_k^T D + Q_k\Big) < 0,
\end{aligned}
\tag{20}
$$

where

$$
\begin{aligned}
\gamma_k &= \min_{1 \le j \le n} (1 - \mu_{kj}), \\
\Delta &= diag\{h_1, h_2, \ldots, h_n\}, \\
Q_k &= diag\{q_{k1}, q_{k2}, \ldots, q_{kn}\} > B_k^T P_k^{-1} B_k.
\end{aligned}
\tag{21}
$$

Proof We construct a Lyapunov functional

$$
\begin{aligned}
V(x(t), t) =& x^T(t)x(t) + 2\alpha \sum_{i=1}^{n} d_i \int_0^{x_i(t)} f_i(s) ds \\
&+ \sum_{k=1}^{N} \sum_{j=1}^{n} \frac{1}{\tau} \int_{-\tau_{kj}(t)}^{0} \int_{t+\theta}^{t} p_{kj} x_j^2(s) ds d\theta \\
&+ \sum_{k=1}^{N} \sum_{j=1}^{n} \Big(\alpha + \frac{1}{1-\mu}\Big) \int_{t-\tau_{kj}(t)}^{t} q_{kj} f_j^2(x_j(s)) ds,
\end{aligned}
$$

where $\mu = \sup_{1 \le k \le N} \{\max_{1 \le j \le n} \mu_{kj}\}$. The remain proof is similar with one of Theorem 1 and we omit it here. □

Illustrative examples

In this section, some examples are demonstrated to show the efficiency of the criteria derived in the Section 3.

Example 1 Consider the delayed neural networks (2) with parameters $C = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, $A = \begin{pmatrix} 1 & -1 \\ -0.5 & 2 \end{pmatrix}$, $B_1 = B_2 = \begin{pmatrix} 0.1 & 0.5 \\ 0.2 & 0.4 \end{pmatrix}$, where $f_1(\vartheta) = \frac{3(|\vartheta+1| - |\vartheta-1|)}{20}$, $f_2(\vartheta) = \frac{(|\vartheta+1| - |\vartheta-1|)}{5}$, $\vartheta \in R$, $\tau_{11}(t) = \tau_{12}(t) = \frac{e^t}{1+e^t}$, $\tau_{21}(t) = \tau_{22}(t) = \frac{t}{2+t}$, $t \ge 0$, $u_1(t) = \sin e^t$, $u_2(t) = \cos e^t$. Obviously, $h_1 = 0.3, h_2 = 0.4$, $i.e. \Delta = \begin{pmatrix} 0.3 & 0 \\ 0 & 0.4 \end{pmatrix}$, $0 \le \dot{\tau}(t) \le 0.5$. Using MATLAB, we can choose

$$
L = \begin{pmatrix} 1.1 & 0 \\ 0 & 0.8 \end{pmatrix}, \quad P_1 = P_2 = \begin{pmatrix} 0.5 & 0 \\ 0 & 1.2 \end{pmatrix},
$$

$$
Q_1 = Q_2 = \begin{pmatrix} 1.1 & 0 \\ 0 & 1.5 \end{pmatrix}.
$$

It follows from Theorem 1 that the delayed neural networks (2) with $N = 2$ are exponentially input-to-state stable. Taking the initial condition $x_1(s) = -0.5$, $x_2(s) = 0.5$, $s \in [-1, 0]$, we give the simulation results of Example 1 in Fig. 1, 2, 3. Figure 1 shows the time responses and Fig. 2 shows the phase graph of the system (2), which illustrates the feature of ISS that its behavior remains bounded when its inputs are bounded for the delayed neural networks. Furthermore, when $u(t) = 0$, Fig. 3 shows the global exponential stability of the origin for the networks.

Example 2 Consider the delayed neural networks (2) with $N = 2$ described by $C = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}$, $A = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}$, $B_1 = B_2 = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.3 \end{pmatrix}$, where $f_1(\vartheta) = tanh\vartheta = \frac{(e^{\vartheta} - e^{-\vartheta})}{(e^{\vartheta} + e^{-\vartheta})}$, $f_2(\vartheta) = \frac{(|\vartheta+1| - |\vartheta-1|)}{2}$, $\vartheta \in R$, $\tau_{ij}(t) = 1 - \frac{1}{2(i+j)} e^{-t}$, $i = 1, 2$, $j = 1, 2$, $t \ge 0$, $u_1(t) = \sin t^5$, $u_2(t) = \cos t^3$. Obviously, $h_1 = h_2 = 1$, $i.e. \Delta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $0 \le \dot{\tau}(t) \le 1$.

Using MATLAB, we can choose D, P_i, Q_i, $i = 1, 2$, as follows:

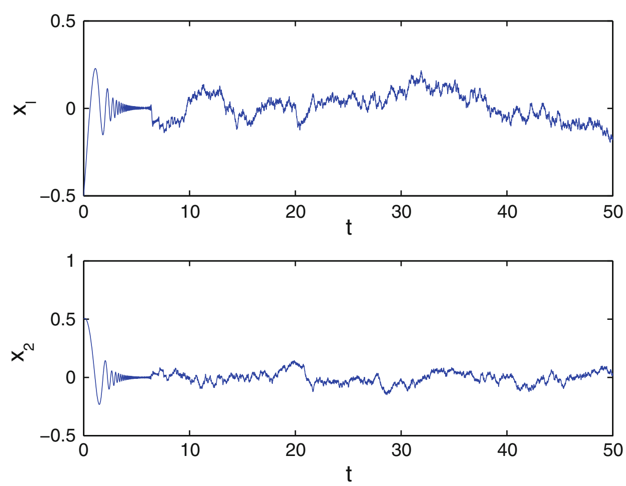

Fig. 1 The time responses of the neural networks in Example 1

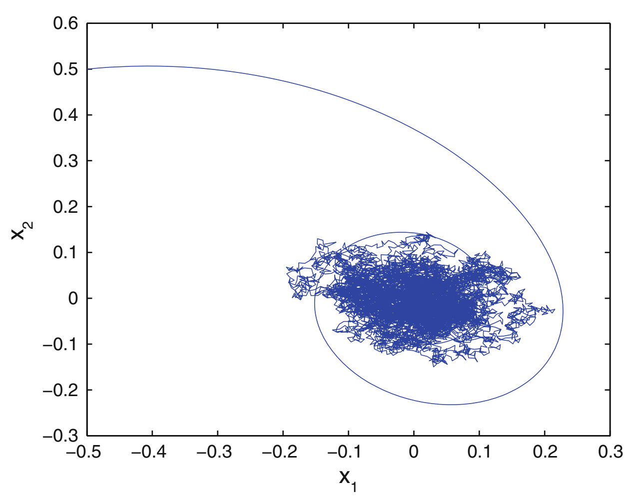

Fig. 2 The phase graph of the neural networks in Example 1

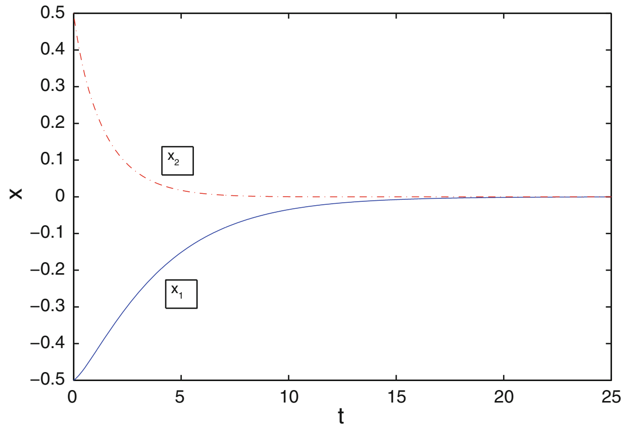

Fig. 3 The time responses of the neural networks with *u(t)* = 0 in Example 1

$$D = \begin{pmatrix} 0.56 & 0 \\ 0 & 0.85 \end{pmatrix}, P_1 = P_2 = \begin{pmatrix} 0.54 & 0 \\ 0 & 0.46 \end{pmatrix}, Q_1 = Q_2 = \begin{pmatrix} 0.08 & 0 \\ 0 & 0.20 \end{pmatrix}.$$

From Theorem 2, we can say that the delayed neural networks (2) with $N = 2$ are exponentially input to stable. Taking the initial condition $x_1(s) = 0.6, x_2(s) = -0.2, s \in [-1, 0]$, we give the simulation results of Example 1 in Fig. 4-6. Figure 4 shows the time responses and Fig. 5 shows the phase graph of the system (2), which illustrates the feature of ISS that its behavior remains bounded when its inputs are bounded for the delayed neural networks. Furthermore, when $u(t) = 0$, Fig. 6 shows the global exponential stability of the origin for the networks.

However, the above results cannot be obtained by using criteria on ISS in existing publications (e.g., Sanchez and Perez 1999; Ahn 2010b, 2011a, b, 2010c; Zhu and Shen 2012).

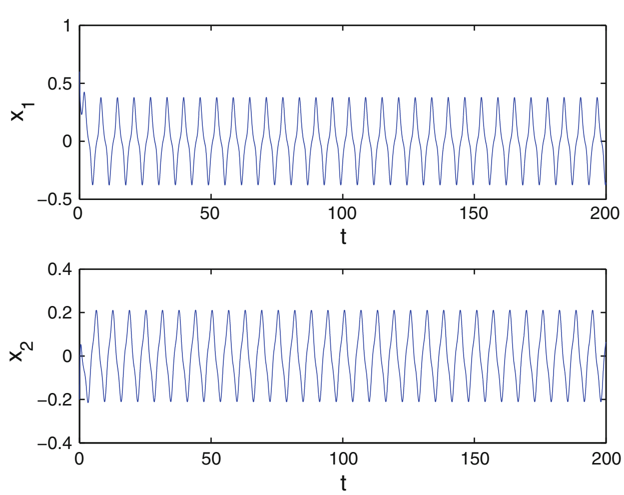

Fig. 4 The time responses of the neural networks in Example 2

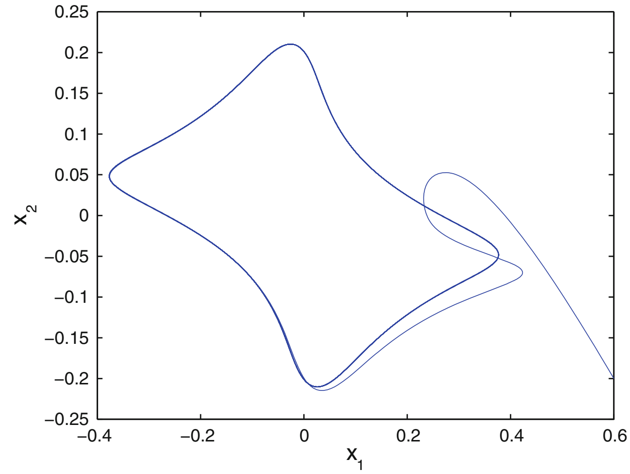

Fig. 5 The phase graph of the neural networks in Example 2

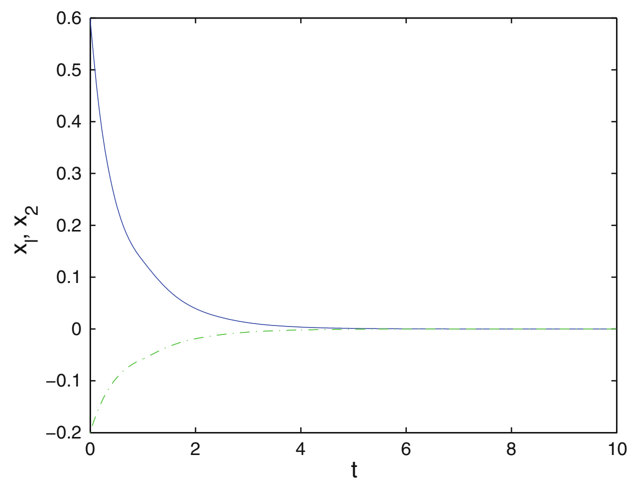

Fig. 6 The time responses of the neural networks with *u(t)* = 0 in Example 2

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