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On the Fiedler vectors of graphs that arise from trees by Schur complementation of the Laplacian

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Abstract

The utility of Fiedler vectors in interrogating the structure of graphs has generated intense interest and motivated the pursuit of further theoretical results. This paper focuses on how the Fiedler vectors of one graph reveal structure in a second graph that is related to the first. Specifically, we consider a point of articulation r in the graph G whose Laplacian matrix is L and derive a related graph $G_{\{r\}}$ whose Laplacian is the matrix obtained by taking the Schur complement with respect to r in L. We show how Fiedler vectors of $G_{\{r\}}$ relate to the structure of G and we provide bounds for the algebraic connectivity of $G_{\{r\}}$ in terms of the connected components at r in G. In the case where G is a tree with points of articulation $r \in R$, we further consider the graph G_R derived from G by taking the Schur complement with respect to R in L. We show that Fiedler vectors of G_R valuate the pendent vertices of G in a manner consistent with the structure of the tree.

1. INTRODUCTION

Let G be a connected weighted graph with vertex set V and edge set E such that each edge is associated with a positive weight. For such a weighted graph, let $A(G) = (a_{ij})$ denote the adjacency matrix of G given by

$$a_{ij}=\begin{cases} \omega & \text{if } (i,j) \in E \text{ and the weight of the edge is } \omega \\ 0 & \text{otherwise} \end{cases}$$

Let $D(G) = (d_{ij})$ be the diagonal matrix with $d_{ii} = \sum_{j \neq i} a_{ij}$. We use $L(G) = D(G) - A(G)$ to denote the Laplacian matrix of G and suppress the G throughout when the context is clear. We follow the convention of using e to denote a conformal vector of ones, and it is clear that $Le = 0$. It is well known that zero is the smallest eigenvalue of L, and when G is connected, as is assumed here, that smallest eigenvalue is simple [1]. The second smallest eigenvalue of L is referred to as the *algebraic connectivity* of the graph G and often denoted μ. In this context, vectors Y satisfying the eigenvalue-eigenvector relationship $LY = \mu Y$ are known as *Fiedler vectors* in honor of the mathematician Miroslav Fiedler [2].

Fiedler first elucidated the properties of the vectors that bear his name, and of primary interest is how such vectors relate back to the graph from which they are derived. Each entry of Y corresponds to a vertex in the graph and Fiedler showed, among other things, that the sets {υ

$\in V : Y(\upsilon) \geq 0\}$ and $\{\upsilon \in V : Y(\upsilon) \leq 0\}$ induce subgraphs of G that are connected. Thus, the sign pattern of a Fiedler vector Y reveals important structure in the graph G, and the body of literature exploiting that fact is extensive. Similarly, theoretical studies of Fiedler vectors and algebraic connectivity abound, including a number of recent works that consider how μ is affected by graph perturbation (e.g. [3]). Much rarer are perturbation results about the Fiedler vectors themselves, possibly because most interesting perturbations change the vertex set of the graph. We elucidate the properties of one such perturbation here.

Our particular interest is Schur complementation. Throughout the text we adopt the standard Schur complement notation of $B/B_{11}=B_{22}-B_{21}B_{11}^{-1}B_{12}$ for a block matrix

$$B=\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}.$$

For any proper subset S of V, we use L_S to denote the principal submatrix of L whose rows and columns are indexed by elements in S. Thus, for a vertex $\upsilon \in V$ corresponding to the m^{th} row and column of L, $L_{V-\{\upsilon\}} = L(m|m)$ and $L_{\{\upsilon\}} = L_{m,m}$. The matrix

$$L/L_{\{\upsilon\}}=L_{V-\{\upsilon\}}-\frac{L_{V-\{\upsilon\}}ee^T L_{V-\{\upsilon\}}}{e^T L_{V-\{\upsilon\}}e} \qquad (1.1)$$

is the Laplacian of a graph that we will denote $G_{\{\upsilon\}}$, and it is this perturbation of G that forms the basis of our study. In Section 3 we build upon our knowledge of $G_{\{\upsilon\}}$ to consider the graph G_S whose Laplacian L/L_S is obtained by removing the vertices in S from G by Schur complementation. To that end, we embark on a study of the graph $G_{\{\upsilon\}}$ whose Laplacian $L/L_{\{\upsilon\}}$ is given in (1.1).

2. PRELIMINARY RESULTS

Recall that a *point of articulation* (or cutpoint) in G is a vertex $r \in V$ whose deletion induces a subgraph $G - r$ with two or more connected components. Let $C_0,\ldots, C_k$ be the connected components at r in G and let L_{C_i} ($i = 0,\ldots, k$) denote the principal submatrix of L whose rows and columns correspond to the vertices in C_i. It is well known that $L_{C_i}^{-1}$ is an entrywise positive matrix whose maximal eigenvalue (henceforth Perron value) $\rho\left(L_{C_i}^{-1}\right)$ is simple. The component C_i at r for which $\rho\left(L_{C_i}^{-1}\right)$ is maximal is called a *Perron component* at r, and we remark that there may be more than one such component [4, 5, 6]. Recent studies have described an intimate relationship between Fiedler vectors and Perron components, and we build upon that line of research below.

The results in this section are general and pertain to arbitrary graphs G. We consider a point of articulation $r \in V$ and study the graph $G_{\{r\}}$ whose Laplacian is $L/L_{\{r\}}$. That $L/L_{\{r\}}$ is a Laplacian is well known [7], and the structure of its graph is intuitive. Specifically, the vertex set of $G_{\{r\}}$ is $V - \{r\}$, and any edges in G that are not incident to r persist in $G_{\{r\}}$. The effect of Schur complementation is to create edges in $G_{\{r\}}$ between any pairs of vertices that are adjacent to r in G. Thus, the structure of $G_{\{r\}}$ is related to the structure of G, and in what follows we describe how Fiedler vectors of $G_{\{r\}}$ relate to the structure of G as well. We begin with a useful lemma that describes how the algebraic connectivity of $G_{\{r\}}$ depends on the connected components at r in G.

Lemma 2.1. *Let G be a connected graph with vertex set V and let $r \in V$ be a point of articulation in G such that the connected components at r are labeled $C_0, C_1, \ldots, C_k$. Let L be the Laplacian of G and let $G_{\{r\}}$ be the graph whose Laplacian is the Schur complement $L/L_{\{r\}}$. Let L_{C_i} denote the principal submatrix of L whose rows and columns correspond to the vertices in C_i, let $\lambda_i = \rho\left(L_{C_i}^{-1}\right)$ be the Perron value of $L_{C_i}^{-1}$, and suppose that the C_i are labeled so that the sequence λ_i, $i = 0, \ldots, k$ is nonincreasing. Let Y_0 and Y_1 be Perron vectors (i.e. eigenvectors of unit norm) corresponding to λ_0 and λ_1. Then the algebraic connectivity μ of $G_{\{r\}}$ satisfies*

$$\left(\frac{(e^T Y_0)^2}{(e^T Y_0)^2+(e^T Y_1)^2}\lambda_1^{-1}+\frac{(e^T Y_1)^2}{(e^T Y_0)^2+(e^T Y_1)^2}\lambda_0^{-1}\right)^{-1} \le \mu^{-1} \le \lambda_0$$

Proof. The Laplacian L can be written as

$$L=\begin{bmatrix} L_{C_0} & 0 & \cdots & 0 & -L_{C_0}e \\ 0 & L_{C_1} & \ddots & \vdots & -L_{C_1}e \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & L_{C_k} & -L_{C_k}e \\ -e^T L_{C_0} & -e^T L_{C_1} & \cdots & -e^T L_{C_k} & \sum_i e^T L_{C_i} e \end{bmatrix}=\begin{bmatrix} B & -Be \\ -e^T B & e^T Be \end{bmatrix}$$

where B represents the upper 4 × 4 block. Each of L_{C_i} is an M-matrix and has an inverse that is positive; hence the Perron vectors Y_0 and Y_1 in the statement of the theorem exist. Moreover, by Perron's Theorem, the entries of Y_0 and Y_1 are positive, so that $e^T Y_0$ and $e^T Y_1$ are nonzero. Consider the vectors Y_0^* and Y_1^* obtained from Y_0 and Y_1 by appending zero entries to each conformally with B:

$$Y_0^*=\begin{bmatrix} Y_0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \text{ and } Y_1^*=\begin{bmatrix} 0 \\ Y_1 \\ \vdots \\ 0 \\ 0 \end{bmatrix}.$$

It is clear that $B^{-1}Y_0^*=\lambda_0 Y_0^*$ and $B^{-1}Y_1^*=\lambda_1 Y_1^*$, and therefore $BY_0^*=\lambda_0^{-1}Y_0^*$ and $BY_1^*=\lambda_1^{-1}Y_1^*$ as well. We introduce the vector Z defined as

$$Z=\frac{(e^T Y_0^*)\,Y_1^* - (e^T Y_1^*)\,Y_0^*}{\sqrt{(e^T Y_0^*)^2+(e^T Y_1^*)^2}}$$

and consider the quadratic form $Z^T BZ$. Notice that by construction Y_0^* and Y_1^* are orthonormal and that $Y_0^{*T} BY_1^*=Y_1^{*T} BY_0^*=0$. Moreover, from the eigenvalue-eigenvector relationship we have that $Y_0^{*T} BY_0^*=\lambda_0^{-1}$ and $Y_1^{*T} BY_1^*=\lambda_1^{-1}$. It follows that

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\begin{aligned}
Z^T BZ &= \frac{\left(\left(e^T Y_0^*\right) Y_1^* - \left(e^T Y_1^*\right) Y_0^*\right)^T B \left(\left(e^T Y_0^*\right) Y_1^* - \left(e^T Y_1^*\right) Y_0^*\right)}{\left(e^T Y_0^*\right)^2 + \left(e^T Y_1^*\right)^2} \\
&= \frac{\left(\left(e^T Y_1^*\right)^2 Y_0^{*T} B Y_0^* + \left(e^T Y_0^*\right)^2 Y_1^{*T} B Y_1^*\right)}{\left(e^T Y_0^*\right)^2 + \left(e^T Y_1^*\right)^2} \\
&= \frac{\left(\left(e^T Y_1^*\right)^2 \lambda_0^{-1} + \left(e^T Y_0^*\right)^2 \lambda_1^{-1}\right)}{\left(e^T Y_0^*\right)^2 + \left(e^T Y_1^*\right)^2} \\
&= \frac{\left(e^T Y_0\right)^2}{(e^T Y_0)^2 + (e^T Y_1)^2} \lambda_1^{-1} + \frac{\left(e^T Y_1\right)^2}{(e^T Y_0)^2 + (e^T Y_1)^2} \lambda_0^{-1}
\end{aligned}
$$

and note that $Z^T Z = 1$ and $e^T Z = 0$ by design. We seek to bound the algebraic connectivity μ of $L/L_{\{r\}}$, and the Courant-Fischer minimax principle tells us that

$$\mu = \min_{X : e^T X = 0} \frac{X^T \ (L/L_{\{r\}}) \ X}{X^T X}.$$

From this representation, it follows immediately that

$$\mu \le \frac{Z^T \ (L/L_{\{r\}}) \ Z}{Z^T Z} = Z^T \left(B - \frac{Bee^T B}{e^T Be}\right) Z = Z^T BZ - \frac{\left(Z^T Be\right) \left(e^T BZ\right)}{e^T Be} \le Z^T BZ$$

where the final inequality holds because $e^T Be = \sum_i e^T L_{C_i} e > 0$. Thus we have

$$\mu \le \frac{\left(e^T Y_0\right)^2}{(e^T Y_0)^2 + (e^T Y_1)^2} \lambda_1^{-1} + \frac{\left(e^T Y_1\right)^2}{(e^T Y_0)^2 + (e^T Y_1)^2} \lambda_0^{-1}$$

and since $\mu > 0$ the left-hand inequality of the lemma is established. The right-hand inequality is an application of Weyl's inequality relating the eigenvalues of B to those of the rank-one perturbation $B - \frac{Bee^T B}{e^T Be}$ (see for example Theorem 1.1 of [8]). By assumption, μ is the second smallest eigenvalue of $B - \frac{Bee^T B}{e^T Be}$ (the smallest, 0, is simple), whereas λ_0^{-1} is the smallest eigenvalue of B. These are guaranteed to interlace so that $0 \le \lambda_0^{-1} \le \mu$. Neither are zero, and so $\mu^{-1} \le \lambda_0$ as desired.

Previous results from Fallat and Kirkland [9] and from Bapat and Pati [10] established deep relationships between the algebraic connectivity of a weighted graph G and the connected components at any of its points of articulation. Our purpose in establishing Lemma 2.1, as revealed in the following series of corollaries, is to show how the connected components at a point of articulation r in G can be used to bound the algebraic connectivity of the derived graph $G_{\{r\}}$.

Corollary 2.1. *Under the conditions of Lemma 2.1, if there is a unique Perron component at r in G, then*

$$\lambda_1 < \mu^{-1} < \lambda_0$$

Proof. That there is a unique Perron component means exactly that $\lambda_1 < \lambda_0$. Both λ_1 and λ_0 are positive, and recall from the proof of Lemma 2.1 that $e^T Y_0 > 0$ and $e^T Y_1 > 0$. Therefore, we have that

$$0 < \frac{\left(e^T Y_0\right)^2}{(e^T Y_0)^2 + (e^T Y_1)^2} < 1.$$

Now, for any c strictly between 0 and 1,

$$\left(c\lambda_1^{-1} + (1-c)\,\lambda_0^{-1}\right)^{-1} = \frac{\lambda_0 \lambda_1}{c\lambda_0 + (1-c)\,\lambda_1}$$

is strictly between λ_1 and λ_0. Thus, λ_1 is strictly less then the lower bound in Lemma 2.1, and we conclude that $\lambda_1 < \mu^{-1}$. To see that $\mu^{-1} < \lambda_0$, suppose $\mu^{-1} = \lambda_0$ by way of contradiction. Then there exists an eigenvector Y of $L/L_{\{r\}}$ with unit norm such that $(L/L_{\{r\}})\, Y = \mu Y$. Writing

$$L/L_{\{r\}} = B - \frac{Bee^T B}{e^T Be}$$

as in the proof of Lemma 2.1, we have that

$$\left(B - \frac{Bee^T B}{e^T Be}\right) Y = \mu Y.$$

Noting that B is invertible, this can be rearranged as

$$B^{-1} Y = \mu^{-1} \left[Y - \left(\frac{e^T BY}{e^T Be}\right) e\right]$$

from which it follows that

$$Y^T B^{-1} Y = \mu^{-1} \left[1 - \left(\frac{e^T BY}{e^T Be}\right) Y^T e\right].$$

On the other hand, because Y is an eigenvector of $L/L_{\{r\}}$ with eigenvalue $\mu > 0$, it is orthogonal to eigenvectors with eigenvalue 0, namely e. Thus $Y^T e = 0$ and we have that $Y^T B^{-1} Y = \mu^{-1} = \lambda_0$. Recall that we have assumed that there is exactly one Perron component at r in G. This implies that the largest eigenvalue of B^{-1}, i.e. λ_0, is simple. Defining Y_0^* as in the proof of Lemma 2.1, it follows that Y_0^* spans the eigenspace of λ_0, and thus our vector Y is a scalar times

Y_0^*. But whereas $e^T Y = 0$, we have seen that $e^T Y_0^*$ is nonzero. This contradiction completes the proof.

Corollary 2.2. *Under the conditions of Lemma 2.1, if there is not a unique Perron component at r in G, then*

$$\lambda_1 = \mu^{-1} = \lambda_0$$

Proof. The assumption that there are at least two Perron components means exactly that $\lambda_0 = \lambda_1$. Substituting this into the bounds of Lemma 2.1 yields the desired result.

Lemma 2.1 and its corollaries bound the algebraic connectivity of $G_{\{r\}}$ in terms of the connected components at r in G and characterize the conditions under which these inequalities are strict. When there is not a unique Perron component at r in G, so that $\lambda_1 = \mu^{-1} = \lambda_0$, the number of Perron components relates to the multiplicity of μ as an eigenvalue of $L/L_{\{r\}}$. To establish this relationship requires use of the following lemma.

Lemma 2.2. *Let B be a symmetric positive definite real matrix with eigenvalues ordered* $\lambda_0 (B) \geq \lambda_1 (B) \geq \cdots \lambda_n (B) > 0$. *Let* $M = B - \frac{Bee^T B}{e^T Be}$ *and order its eigenvalues* $\lambda_0 (M) \geq \lambda_1 (M) \geq \cdots \lambda_n (M) = 0$.. *Suppose that* $\lambda_n (B) = \lambda_{n-1} (M)$, *and let Y be an eigenvector of M with eigenvalue* $\lambda_{n-1} (M)$. *Then Y is an eigenvector of B as well.*

Proof. Let Y be as described in the statement of the lemma and suppose without loss of generality that $Y^T Y = 1$. We have that $MY = \lambda_n (B) Y$ and so

$$\left(B - \frac{Bee^T B}{e^T Be}\right) Y = \lambda_n (B) Y \quad (2.1)$$

as well. Noting that by assumption B is invertible, (2.1) can be rearranged as

$$B^{-1} Y = \lambda_n (B)^{-1} \left[Y - \left(\frac{e^T BY}{e^T Be}\right) e \right]$$

from which it follows that

$$Y^T B^{-1} Y = \lambda_n (B)^{-1} \left[1 - \left(\frac{e^T BY}{e^T Be}\right) Y^T e \right]. \quad (2.2)$$

Because Y is an eigenvector of M with eigenvalue $\lambda_{n-1} (M)$, the Courant-Fischer minimax principle guarantees that Y is orthogonal to eigenvectors of M with eigenvalue $\lambda_n (M)$, namely e. Thus $Y^T e = 0$ and (2.2) becomes $Y^T B^{-1} Y = \lambda_n (B)^{-1}$. Now $\lambda_n (B)^{-1}$ is the largest eigenvalue of B^{-1}, and so $Y^T B^{-1} Y = \lambda_n (B)^{-1}$ implies that $B^{-1} Y = \lambda_n (B)^{-1} Y$ as well. Hence $BY = \lambda_n (B) Y$, completing the proof of the lemma.

In the context of Lemma 2.1, we have now established that in the case where there are two or more Perron components at r in G, the Fiedler vectors of $L/L_{\{r\}}$ are also eigenvectors corresponding to eigenvalues of $L_{V-\{r\}}$ of maximum modulus. This has interesting implications. Suppose that there are $m \geq 2$ Perron components at r in G. We know from Corollary 2.2 that $\lambda_1 = \mu^{-1} = \lambda_0$, and it is clear that this implies $\lambda_0 = \lambda_1 = \cdots = \lambda_{m-1} =$

μ^{-1}. Let $Y_0, Y_1, \ldots, Y_{m-1}$ denote the corresponding Perron vectors, and define $Y_0^*, Y_1^*, \ldots, Y_{m-1}^*$ as in the proof of Lemma 2.1. The vectors $Y_0^*, Y_1^*, \ldots, Y_{m-1}^*$ span the eigenspace of λ_0 in $L_{\{r\}}$ and thus *span* $(\{Y_0^*, Y_1^*, \ldots, Y_{m-1}^*)\}$ contains the eigenspace of μ in $L/L_{\{r\}}$. On the other hand, it easy to verify that the eigenspace of μ in $L/L_{\{r\}}$ contains all linear combinations of $Y_0^*, Y_1^*, \ldots, Y_{m-1}^*$ that are orthogonal to the vector e. From this we deduce that the eigenspace of μ in $L/L_{\{r\}}$ is exactly $\{\upsilon\colon\ (L/L_{\{r\}})\ \upsilon=\mu\upsilon\}=\left\{\upsilon=\sum_{i=0}^{m-1} a_i Y_i^* : a_0, \ldots, a_{m-1} \in \mathbb{R} \text{ and } e^T\upsilon=0\right\}$ $\sum_{i=0}^{m-1} a_i Y_i^*$. The dimension of this space is evidently $m - 1$, one less than the number of Perron components. We collect these results in a proposition for future reference.

Proposition 2.1. *Suppose that there are $m \geq 2$ Perron components at r in G, and let $Y_0^*, Y_1^*, \ldots, Y_m^*$ be as defined in the proof of Lemma 2.1. Then the Fiedler vectors Y of $L/L_{\{r\}}$ satisfy $e^TY = 0$ and can be written as* $\sum_{i=0}^{m-1} a_i Y_i^*$ for some scalars $a_0, \ldots, a_{m-1} \in \mathbb{R}$. *Conversely, for any scalars $a_0, \ldots, a_{m-1} \in \mathbb{R}$, if $Y=\sum_{i=0}^{m-1} a_i Y_i^*$ satisfies $e^TY = 0$ then it is a Fiedler vector of $L/L_{\{r\}}$.*

3. MAIN RESULTS

In the previous section we described the relationship between Fiedler vectors of $G_{\{r\}}$ and the connected components at r in G. In this section we make the additional assumptions that r is adjacent to every other vertex in V and that the connected components at r in G are complete. As the following theorem shows, these further restrictions guarantee that the sign pattern of a Fiedler vector Y of $G_{\{r\}}$ can be used to cut the graph G. The proof indicates where the assumptions can be weakened without invalidating its result.

Theorem 3.1. *Let G be a connected graph with vertex set V and let $r \in V$ be a point of articulation in G adjacent to every other vertex in V and such that the connected components at r are complete. Let L be the Laplacian of G and let $G_{\{r\}}$ be the graph whose Laplacian is the Schur complement $L/L_{\{r\}}$. For any Fiedler vector Y of $L/L_{\{r\}}$ corresponding to the algebraic connectivity μ of $G_{\{r\}}$, let $V_+ = \{\upsilon \in V - \{r\} : Y(\upsilon) > 0\}$ and let V_- be its complement in $V - \{r\}$. Then there exists a Fiedler vector Y such that (1) both V_+ and $V_- \cup \{r\}$ induce connected subgraphs in G or (2) both V_- and $V_+ \cup \{r\}$ induce connected subgraphs in G.*

Proof. The proof hinges on the structure of the graph G. As in the proof of Lemma 2.1, enumerate the connected components at r as $C_0, C_1, \ldots, C_k$ and permute the Laplacian of G so that it can be written as

$$L=\begin{bmatrix} L_{C_0} & 0 & \cdots & 0 & -L_{C_0}e \\ 0 & L_{C_1} & \ddots & \vdots & -L_{C_1}e \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & L_{C_k} & -L_{C_k}e \\ -e^TL_{C_0} & -e^TL_{C_1} & \cdots & -e^TL_{C_k} & \sum_i e^TL_{C_i}e \end{bmatrix}=\begin{bmatrix} B & -Be \\ -e^TB & e^TBe \end{bmatrix}$$

where B represents the upper 4 × 4 block. Let λ_i once again denote the Perron value $\rho\left(L_{C_i}^{-1}\right)$ of the entrywise positive matrix $L_{C_i}^{-1}$, and without loss of generality assume the sequence λ_i, $i = 0, \ldots, k$ is nonincreasing. We are interested in the matrix $L/L_{\{r\}}$, which in terms of B can be written as

$$L/L_{\{r\}}=B-\frac{Bee^TB}{e^TBe}$$

Now consider the Fiedler vector Y of $L/L_{\{r\}}$ and partition it conformally as

$$Y=\begin{bmatrix} Y_0 \\ Y_1 \\ \vdots \\ Y_n \end{bmatrix}.$$

The eigenvector-eigenvalue relationship gives $(L/L_{\{r\}})\ Y = \mu Y$, or in other words

$$\left(B-\frac{Bee^TB}{e^TBe}\right)\ Y=\mu Y$$

which can be rewritten as

$$(B^{-1}-\mu^{-1}I)\ Y=\left(\mu^{-1}\frac{e^TBY}{e^TBe}\right)\ e.$$

The right-hand side of this equation is a scalar times e, while the left-hand can be expanded as

$$(B^{-1}-\mu^{-1}I)\ Y=\begin{bmatrix} (L_{C_0}^{-1}-\mu^{-1}I)\ Y_0 \\ (L_{C_1}^{-1}-\mu^{-1}I)\ Y_1 \\ \vdots \\ (L_{C_k}^{-1}-\mu^{-1}I)\ Y_k \end{bmatrix}.$$

Thus we have for each $i = 0,\dots, k$ that $\left(L_{C_i}^{-1}-\mu^{-1}I\right)\ Y_i=\left(\mu^{-1}\frac{e^TBY}{e^TBe}\right)\ e$.

First consider the case where there is a unique Perron component at r in $G_{\{r\}}$. Corollary 2.1 guarantees that $\lambda_1 < \mu^{-1}$, and since λ_1 is the largest of the eigenvalues of $L_{C_i}^{-1}$ it is clear that μ^{-1} cannot be an eigenvalue of $L_{C_i}^{-1}$ for any $i \geq 1$. This implies that $L_{C_i}^{-1}-\mu^{-1}I$ is invertible, which allows us to solve for Y_i, $i \geq 1$ as

$$Y_i=(L_{C_i}^{-1}-\mu^{-1}I)^{-1}\ \left(\mu^{-1}\frac{e^TBY}{e^TBe}\right)\ e.$$

Next, note that $\mu^{-1}I-L_{C_i}^{-1}$ is by definition an M-matrix since $L_{C_i}^{-1}$ is a positive matrix whose spectral radius (Perron value) is strictly less than μ^{-1} (see, for example [11]). Thus, the matrix $\mu^{-1}I-L_{C_i}^{-1}$ is inverse positive, meaning that the entries of $(\mu^{-1}I-L_{C_i}^{-1})^{-1}$ are strictly greater than zero. This implies that the entries of $(L_{C_i}^{-1}-\mu^{-1}I)^{-1}$ are negative for each $i \geq 1$. Thus, the

entries of $(L_{C_i}^{-1} - \mu^{-1} I)^{-1} e$ are negative, which proves that each of the entries of Y_i is the opposite sign of $e^T BY$ (since μ^{-1} and $e^T Be$ are positive). First suppose that $e^T BY$ is positive, so that none of the vertices of C_i, $i \geq 1$ are contained in V_+. Then V_+ contains only vertices in C_0, and since C_0 is complete, V_+ induces a connected subgraph in G. Because r has an edge incident to every vertex, $V_- \cup \{r\}$ must be connected as well. Conversely, if $e^T BY$ is negative, it is V_- that contains none of the vertices of C_i, $i \geq 1$. Again, because C_0 is complete, V_- induces a connected subgraph in G. That $V_+ \cup \{r\}$ is connected follows as before.

Thus, the theorem holds in the case where there is a unique Perron component, and we are left to consider the alternative. Suppose that there are $m \geq 2$ Perron components at r in $G_{\{r\}}$ labelled $C_0, C_1, \ldots, C_{m-1}$ as before. Let $Y_0, Y_1, \ldots, Y_{m-1}$ be Perron vectors corresponding respectively to the $L_{C_i}^{-1}$, and define $Y_0^*, Y_1^*, \ldots, Y_{m-1}^*$ as in the proof of Lemma 2.1. Thus, as before, for $0 \leq i \leq m-1$, Y_i^* valuates the vertices of C_i as positive and valuates all other vertices as zero.

Proposition 2.1 characterizes the Fiedler vectors of $L/L_{\{r\}}$ as vectors $Y = \sum_{i=0}^{m-1} a_i Y_i^*$, $a_0, \ldots, a_{m-1} \in \mathbb{R}$ that satisfy $e^T Y = 0$. Therefore, in light of how the $Y_0^*, Y_1^*, \ldots, Y_{m-1}^*$ were constructed, any vertices $v \in V - \cup_{i=0}^{m-1} C_i$ (i.e. those that do not belong to one of the m Perron components at r in G) are valuated as zero by all Fiedler vectors of $L/L_{\{r\}}$. The remaining vertices can be valuated as either positive or negative depending on the choice of Fiedler vector; however, vertices from the same connected component in G are always valuated with the same sign. To complete the proof, we simply choose a Fiedler vector whose positive valuations are restricted to exactly one Perron component, say C_0. Consider, for example, the vector

$$Y \triangleq \frac{\left(\sum_{i=1}^{m-1} e^T Y_i^*\right) Y_0^* - (e^T Y_0^*) \sum_{i=1}^{m-1} Y_i^*}{\sqrt{\left(\sum_{i=1}^{m-1} e^T Y_i^*\right)^2 + (e^T Y_0^*)^2}}$$

which is clearly a Fiedler vector of $L/L_{\{r\}}$ by Proposition 2.1. Note that $\sum_{i=1}^{m-1} e^T Y_i^* > 0$ so that $\left(\sum_{i=1}^{m-1} e^T Y_i^*\right) Y_0^*$ valuates the vertices of C_0 as positive and valuates all remaining vertices as zero. In addition, $e^T Y_0^* > 0$ and the vector $\sum_{i=1}^{m-1} Y_i^*$ valuates the vertices of C_i, $1 \leq i \leq m-1$ as positive and valuates all remaining vertices as zero. Taken together, the vector Y is positive on C_0, negative on C_i, $1 \leq i \leq m-1$, and zero elsewhere. We conclude that V_+ contains only vertices in C_0, which as above implies that both V_+ and $V_- \cup \{r\}$ are connected.

We now specialize to the case where the graph G is a tree. As such, the vertex set V can be partitioned into points of articulation R and pendent vertices P. Recall that for any $S \subset V$, L_S denotes the principal submatrix of L that corresponds to the vertices in S. If S is a proper subset of the vertex set V of G, the matrix L/L_S can also be viewed as a Laplacian, and we call its graph G_S. In light of the discussion in Section 2, we can make the following claim:

Claim 3.2. *Let G be a tree and let V be its vertex set. Let S be a proper subset of V and let G_S be the graph whose Laplacian matrix is L/L_S. Then there is an edge in G_S connecting $i,j \in V - S$ if and only if either (1) i and j are adjacent in G or (2) there is a path from i to j in G that only traverses vertices in S.*

When $R \subset S$ so that S includes all of the points of articulation of the tree G, it follows that G_S is complete. We focus on the case $R = S$, as the graph G_R holds special interest. In particular,

we can obtain the graph G_R from the graph $G_{R-\{r\}}$ by Schur complementation for any point of articulation $r \in R$ in G, and $G_{R-\{r\}}$ retains some structure of the tree G at the vertex r. Specifically, the graph $G_{R-\{r\}}$ contains exactly one point of articulation, namely r. Moreover, the connected components at r in $G_{R-\{r\}}$ are complete, and r is adjacent to each of the vertices in $V - R$. Thus, $G_{R-\{r\}}$ is of exactly the structure required to invoke Theorem 3.1. Therefore, if Y is a Fiedler vector corresponding to the graph G_R that valuates the vertices in $V - R$, it follows that either (1) both V_+ and $V_- \cup \{r\}$ induce connected subgraphs in $G_{R-\{r\}}$ or (2) both V_- and $V_+ \cup \{r\}$ induce connected subgraphs in $G_{R-\{r\}}$. Crucially, this is true for *every* $r \in R$, and as the following theorem shows, a consequence is that meaningful structure in G can be recovered.

Theorem 3.3. *Let G be a tree with vertex set V and let $R \subset V$ be the set of points of articulation in G. Let L be the Laplacian of G and let G_R be the graph whose Laplacian is the Schur complement L/L_R. Let Y be a Fiedler vector of L/L_R corresponding to the algebraic connectivity μ of G_R. Let $V_+ = \{\upsilon \in V - R : Y(\upsilon) > 0\}$ and let V_- be its complement in $V - R$. Then there exists a subset S of R such that both $V_+ \cup S$ and $V_- \cup (R - S)$ induce connected subgraphs in G.*

Proof. Our proof is constructive. Partition the vertex set V of G into two sets R and P containing points of articulation and pendent vertices, respectively, and note that the vertex set of G_R is $V - R = P$. Index the vertices in R as r_k arbitrarily and consider the graphs $G_{R-\{r_k\}}$. We remark for the sake of clarity that the vertices in V_+ and V_- are pendent in G, that the vertex set of $G_{R-\{r_k\}}$ is $P \cup \{r_k\}$, and that $V_+ \cup V_- = P$. Now from Theorem 3.1, we know that either (1) V_+ and $V_- \cup \{r_k\}$ induce connected subgraphs in $G_{R-\{r_k\}}$ or (2) V_- and $V_+ \cup \{r_k\}$ induce connected subgraphs in $G_{R-\{r_k\}}$. Let $S = \{r_k : V_-$ and $V_+ \cup \{r_k\}$ are connected in $G_{R-\{r_k\}}\}$. We claim that $V_+ \cup S$ and $V_- \cup (R - S)$ induce connected subgraphs in G

Let $a, b \in V_+ \cup S$. Because G is a tree, there exists a unique path from a to b in G. To show that $V_+ \cup S$ is connected in G, we must prove that each vertex along that path belongs to $V_+ \cup S$ as well. If a and b are adjacent in G, we are done. Otherwise, let υ be any vertex distinct from a and b that lies on the path between them. By definition, the vertex υ is not pendent, and so we may assume $\upsilon \in R$. Since $V_+ \subset V - R = P$, $\upsilon \notin V_+$, and to prove that $\upsilon \in V_+ \cup S$ requires us to show that $\upsilon \in S$. We assume by way of contradiction that $\upsilon \in R - S$.

We proceed in cases, considering first the case where $a, b \in V_+$. Because $\upsilon \in R - S$, V_+ is connected in $G_{R-\{\upsilon\}}$, which by Claim 3.2 is possible only if V_+ is contained in exactly one connected component at υ in G. But for υ to lie on the path between a and b in G means that a and b fall in distinct connected components, contradicting our assumption that $\upsilon \in R - S$. We conclude that if $a, b \in V_+$, then the path between a and b in G is comprised exclusively of vertices in S.

Next, suppose that $a \in V_+$ and $b \in S$. Again, because $\upsilon \in R - S$, V_+ is connected in $G_{R-\{\upsilon\}}$, and so V_+ must be contained in exactly one connected component at υ in G, in this case the one that includes the vertex a. The vertex $b \in S$ is in a distinct connected component, and all of the pendent vertices in G that belong to this connected component must therefore belong to V_-. Now because b is a point of articulation, there must be at least one pendent vertex $x \in V_-$ in this connected component whose path in G from x to υ contains b. On the other hand, because $\upsilon \in R - S$, V_- is not connected in $G_{R-\{\upsilon\}}$, and so there must exist $y \in V_-$ in a connected component at υ in G that is distinct from the one in which x resides. In particular, by construction the path from x to y in G contains b; however, because $b \in S$ it must be that V_- is connected in $G_{R-\{b\}}$. These two statements are contradictory, and so it cannot be that $\upsilon \in R - S$.

Finally, suppose that $a, b \in S$. Once more, because $\upsilon \in R - S$, V_+ is contained in exactly one connected component at υ in G. As υ is on the path from a to b, at least one of a and b belongs to a connected component at υ in G whose pendent vertices are exclusively in V_-. We can assume this to be b without loss of generality, at which point the argument from the previous case applies.

Having reached a contradiction in every case, we conclude that $\upsilon \notin R - S$, from which it follows that $V_+ \cup S$ is connected in G. The proof that $V_- \cup (R - S)$ is connected in G is similar.

Theorems 3.1 and 3.3 are implicitly results about splits, a concept we now formally define. Let $X \subset V$ and let X_1 and X_2 be nonempty disjoint subsets of X such that $X = X_1 \cup X_2$. We call $\{X_1, X_2\}$ a *split* of G if there exists a partition $\{Z_1, Z_2\}$ of $V - X$ such that both $X_1 \cup Z_1$ and $X_2 \cup Z_2$ induce subgraphs of G that are connected. Theorem 3.1 shows that, under certain conditions, the signs of the entries of Fiedler vectors corresponding to the graph $G_{\{r\}}$ are guaranteed to induce splits of the parent graph G. Theorem 3.3 goes further in the case where G is a tree. If G is a tree, $\{X_1, X_2\}$ is a split in G if and only if there exists at least one edge in E that belongs to every path between arbitrary vertices $x_1 \in X_1$ and $x_2 \in X_2$. When $X = P$, each split corresponds to exactly one edge in this manner, and vice versa. Theorem 3.3 considers this case and concludes that Fiedler vectors corresponding to the graph G_R can be used to construct bipartitions of P that are splits of G. In short, the valuations of P from Fiedler vectors of L/L_R identify an edge in G by their signs.

4. APPLICATION

Theorem 3.3 is the main result of the paper and has a variety of applications. Our application in this section is to a tree G whose Laplacian has not been directly supplied. As such, let G be a weighted tree with vertex set V and edge set E, and partition V into pendent vertices P and points of articulation R. We suppose that the Laplacian L of G has not been supplied, but that the Laplacian L/L_R of G_R has been supplied instead. The graph G_R is complete and the entries of L/L_R are strictly nonzero; nevertheless, Theorem 3.3 guarantees that Fiedler vectors of L/L_R reveal structure in the tree G.

To illustrate the point and our approach in general, consider a tree G with vertex set $V = \{1, \ldots, 8\}$ whose Laplacian L is given below:

$$L=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ -1 & -2 & 0 & 0 & 0 & 4 & 0 & -1 \\ 0 & 0 & -3 & -2 & 0 & 0 & 6 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 & -1 & 3 \end{bmatrix}.$$

The points of articulation in G are $R = \{6, 7, 8\}$, while the pendent vertices are $P = \{1, 2, 3, 4, 5\}$. We suppose that L has not been supplied and consider the graph G_R whose Laplacian L/L_R is

$$L/L_R=\begin{bmatrix} 0.7258 & -0.5484 & -0.0484 & -0.0323 & -0.0968 \\ -0.5484 & 0.9032 & -0.0968 & -0.0645 & -0.1935 \\ -0.0484 & -0.0968 & 1.4032 & -1.0645 & -0.1935 \\ -0.0323 & -0.0645 & -1.0645 & 1.2903 & -0.1290 \\ -0.0968 & -0.1935 & -0.1935 & -0.1290 & 0.6129 \end{bmatrix}.$$

The algebraic connectivity of G_R, $\mu = 0.3836$, is a simple eigenvalue of L/L_R. Its corresponding Fiedler vector,

$$Y^T=[0.57920.4418\ -0.4641\ -0.5008\ -0.0562] \quad (4.1)$$

identifies the split $\{V_+, V_-\} = \{\{1, 2\}, \{3, 4, 5\}\}$ of G, which corresponds to the edge that is incident to 6 and 8 in G. The cut that corresponds to this split partitions the vertices into $V_+ \cup S = \{1, 2, 6\}$ and $V_- \cup (R - S) = \{3, 4, 5, 7, 8\}$, where $S = \{6\}$ can be constructed as in the proof of Theorem 3.3. The proof of the theorem exploits the relationship between G_R and the three graphs $G_{\{6,7\}}$, $G_{\{6,8\}}$, and $G_{\{7,8\}}$ (see Figure 1), and as we have shown, the structure of each of these graphs places bounds on μ and constrains the Fiedler vector Y in some way. For example, in the context of Lemma 2.1, the three connected components $C_0 = \{1, 2\}$, $C_1 = \{3, 4\}$, $C_2 = \{5\}$ at 8 in $G_{\{6,7\}}$ identify three principal submatrices of $L/L_{\{6,7\}}$ whose inverses have Perron values $\lambda_0 = 2.7808$, $\lambda_1 = 2.4201$ and $\lambda_2 = 1$, respectively. The lemma, along with Corollaries 2.1 and 2.2, tells us that μ is a simple eigenvalue of L/L_R bounded strictly between $\lambda_0^{-1}=0.3596$ and $\lambda_1^{-1}=0.4132$. Theorem 3.1, on the other hand, asserts that the Fiedler vector Y must use the same sign to valuate the vertices in $C_1 \cup C_2 = \{3, 4, 5\}$, but it says nothing about the vertices in the Perron component C_0 at 8 in $G_{\{6,7\}}$. For this information we turn to $G_{\{7,8\}}$, whose Perron component at 6 is $\{3, 4, 5\}$. Theorem 3.1 now says that Y must use the same sign to valuate the vertices in $P - \{3, 4, 5\} = \{1, 2\}$, which resolves the sign pattern in (4.1) completely. Because $G_{\{6,7\}}$, $G_{\{6,8\}}$, and $G_{\{7,8\}}$ collectively carry the structure of the tree G, Theorem 3.3 guarantees this sign pattern will induce a split.

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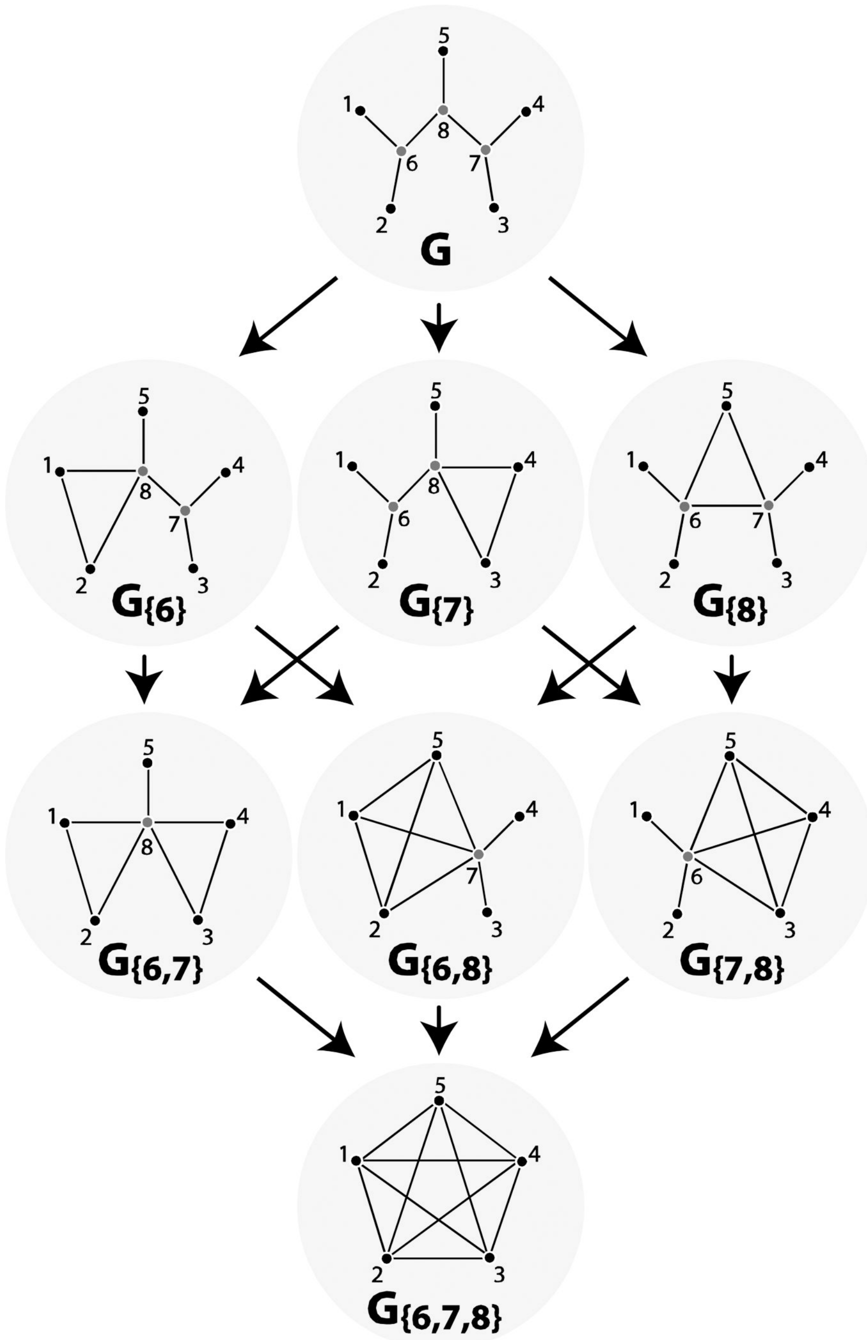

Fig. 1.
The Schur hierarchy linking G and $G_{\{6,7,8\}}$. Arrows are drawn to distinguish pairs of graphs before and after the removal of one vertex by Schur complementation.